39. On Fundamental Solution of Differential Equation with Time Delay in Banach Space

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This paper is concerned with the fundamental solution in the sense of S. Nakagiri [4] to the linear differential equation with time delay

$$\frac{d}{dt}u(t)=Au(t)+A_1u(t-r)+\int_{-r}^{0}a(\tau)A_2u(t+\tau)d\tau+f(t) \tag{1}$$

in a Banach space X. We assume that

(i) A is a densely defined closed linear operator which generates an analytic semigroup $T(t)$ in X.

(ii) A_1 and A_2 are closed linear in general unbounded operators with domains $D(A_1)$ and $D(A_2)$ containing the domain $D(A)$ of A.

(iii) a is a uniformly Hölder continuous real valued function in $[-r,0]$, where r is some fixed positive number.

For the sake of convenience we assume that A has an everywhere defined bounded inverse.

The solvability of the initial value problem for the equation (1) as well as fundamental results on the semigroup associated with it was established by G. Di Blasio, K. Kunisch, and E. Sinestrary [1], [2], [3], [11] under a mild smoothness hypothesis on the coefficient a in the delay term, i.e. $a\in L^1(-r,0)$ or $a\in L^2(-r,0)$.

The fundamental solution $W(t)$ to the equation (1) is by definition a bounded operator valued function satisfying

$$W(t)=\begin{cases}T(t)+\int_0^t T(t-s)\{A_1W(s-r)+\int_{-r}^{0}a(\tau)A_2W(s+\tau)d\tau\}ds, & t\geqq 0\\ 0 & t<0.\end{cases}$$

With the aid of the change of the variable $\tau\to\tau-s$ and noting that $W(t)=0$ for $t<0$ we get

$$W(t)=T(t)+\int_0^t T(t-s)\int_0^s a(\tau-s)A_2W(\tau)d\tau ds \tag{2}$$

in $[0,r]$, and

$$\begin{aligned}W(t)=T(t)&+\int_r^t T(t-s)A_1W(s-r)ds\\ &+\int_0^t T(t-s)\int_{s-r}^{s}a(\tau-s)A_2W(\tau)d\tau ds\end{aligned} \tag{3}$$

in (r,∞). The exchange of the order of integration yields

$$W(t)=T(t)+\int_0^t\int_\tau^t T(t-s)a(\tau-s)dsA_2W(\tau)d\tau \tag{4}$$

in $[0,r]$, and