37. Algebraic Equations for Green Kernel on a Tree

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Let Γ be a connected, locally finite tree with the set of vertices $V(\Gamma)$. Let A be a symmetric operator on $l^2(\Gamma)$, the space of square summable complex valued functions on $V(\Gamma)$:

$$(1)\qquad Au(\gamma)=\sum_{\langle\gamma',\gamma\rangle} a_{\gamma,\gamma'}u(\gamma')+a_{\gamma,\gamma}u(\gamma),$$

for $u\in l^2(\Gamma)$, with $a_{\gamma,\gamma}$ and $a_{\gamma,\gamma'}\in \boldsymbol{R}$ such that $a_{\gamma,\gamma'}\neq 0$, where $\langle\gamma,\gamma'\rangle$ means that γ and γ' are adjacent to each other. We assume that A is self-adjoint with the domain $\mathcal{D}(A)$: $\{u\in l^2(\Gamma)\,|\,\sum_{\gamma\in V(\Gamma)}|u(\gamma)|^2<\infty\}$. Then there exists uniquely the Green function $G(\gamma,\gamma'|z)$ for A, $\gamma,\gamma'\in V(\Gamma)$, representing the resolvent $(z-A)^{-1}$ for $z\in \boldsymbol{C}$, $\mathrm{Im}\, z\neq 0$:

$$(2)\qquad G(\gamma,\gamma'|z)=\int_{-\infty}^{+\infty}\frac{d\Theta(\gamma,\gamma'|\lambda)}{z-\lambda}$$

for the spectral kernel $\Theta(\gamma,\gamma'|\lambda)$ of A. We remark that for any $\gamma\in V(\Gamma)$, $G(\gamma,\gamma|z)$ satisfies

$$(3)\qquad \mathrm{Im}\, G(\gamma,\gamma|z)\cdot \mathrm{Im}\, z<0.$$

The purpose of this note is to extend a result obtained in [3] and [4] to an arbitrary tree. Algebraicity of Green functions was proved under various contexts. Here we want to give explicit formulae for them for an arbitrary self adjoint operator (see [3], [8] and [9]). First we want to prove

Lemma 1. *For arbitrary adjacent vertices γ,γ', suppose γ' and $\gamma_0\in V(\Gamma)$ do not lie in the same connected component of $\Gamma-\{\gamma\}$. Then the quotient $G(\gamma_0,\gamma'|z)/G(\gamma_0,\gamma|z)$ does not depend on γ_0.*

Proof. We denote by $\Gamma_{\gamma'}$ the connected subtree of Γ consisting of vertices γ'' lying in the connected component containing γ' of $\Gamma-\{\gamma\}$. We consider the following boundary value problem on the connected subtree $\Gamma_{\gamma'}\cup\{\gamma\}$ containing $\Gamma_{\gamma'}$ and γ: To find a solution $u\in l^2(\Gamma_{\gamma'}\cup\{\gamma\})$ such that

$$(4)\qquad Au(\gamma'')=zu(\gamma'')\qquad \text{for } \gamma''\in V(\Gamma_{\gamma'}),$$

$$(5)\qquad u(\gamma)=1.$$

Then every $G(\gamma_0,\gamma''|z)/G(\gamma_0,\gamma|z)$ is a solution for this problem. Hence Lemma 1 follows from the following:

Lemma 2. *There exists the unique solution $u(\gamma'')$ for the problem* (4) *and* (5).

Proof. Suppose that there exist two solutions $u_1(\gamma'')$ and $u_2(\gamma'')$ on $V(\Gamma_{\gamma'}\cup\{\gamma\})$. Then the difference $v=u_1-u_2$ also satisfies (4) and vanishes at γ. We have to prove that v vanishes identically. We define a function $\tilde{v}$ on $V(\Gamma)$ such that

$$(6)\qquad \tilde{v}(\gamma'')=v(\gamma'')\qquad \text{for } \gamma''\in V(\Gamma_{\gamma'}),$$