24. On Cayley-Hamilton's Theorem and Amitsur-Levitzki's Identity

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1. The purpose of this note is to prove a generalization of the classical Cayley-Hamilton's theorem and a tensor version of Amitsur-Levitzki's identity concerning matrices.

Let V be an n-dimensional vector space over the field of complex numbers and $A_1, \cdots, A_p$ be linear endomorphisms of V. We define a linear map $A_1 \wedge \cdots \wedge A_p : \wedge^p V \longrightarrow \wedge^p V$ ($\wedge^p V$ is the skew symmetric tensor product of V) by

$$(A_1 \wedge \cdots \wedge A_p)(u_1 \wedge \cdots \wedge u_p) = (1/p!) \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma A_1 u_{\sigma(1)} \wedge \cdots \wedge A_p u_{\sigma(p)},$$

where $(-1)^\sigma$ is the signature of the permutation $\sigma \in \mathfrak{S}_p$ and $u_1, \cdots, u_p \in V$. Note that the equality $A_{\sigma(1)} \wedge \cdots \wedge A_{\sigma(p)} = A_1 \wedge \cdots \wedge A_p$ holds for any permutation $\sigma \in \mathfrak{S}_p$. For $X \in \mathrm{End}(V)$, we define invariants $f_i(X) \in \boldsymbol{C}$ by

$$\det(\lambda I - X) = \sum_{i=0}^n f_i(X) \lambda^{n-i},$$

where I is the identity matrix. Then we have

Theorem 1. *Let X be a linear endomorphism of V and p be an integer $(1 \leqq p \leqq n)$. Then, by putting $r = n+1-p$, the following identity holds:*

$$(1)\quad \begin{aligned} &\sum_{\substack{a_1+\cdots+a_p=r \\ a_i \geq 0}} X^{a_1} \wedge \cdots \wedge X^{a_p} + f_1(X) \sum_{\substack{a_1+\cdots+a_p=r-1 \\ a_i \geq 0}} X^{a_1} \wedge \cdots \wedge X^{a_p} + \cdots \\ &+ f_{r-1}(X) \sum_{\substack{a_1+\cdots+a_p=1 \\ a_i \geq 0}} X^{a_1} \wedge \cdots \wedge X^{a_p} + f_r(X) \cdot \overbrace{I \wedge \cdots \wedge I}^{p} = 0 : \\ &\qquad \wedge^p V \longrightarrow \wedge^p V, \end{aligned}$$

where the sum is taken over all the combinations of integers $\{a_i\}$ satisfying the conditions under Σ. (We consider $X^0 = I$.)

Remark. In the case $p=1$, the above identity is reduced to the form:

$$X^n + f_1(X) X^{n-1} + \cdots + f_n(X) \cdot I = 0 : V \longrightarrow V,$$

which is nothing but the classical Cayley-Hamilton's theorem.

Proof. We have only to prove the theorem in case where X is a diagonal matrix because such a matrix constitutes a dense subset of the space of matrices. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the eigenvalues of X and $\{e_1, \cdots, e_n\}$ be a basis of V such that $X e_i = \alpha_i e_i$. We prove that the element $e_1 \wedge \cdots \wedge e_p \in \wedge^p V$ is mapped to 0 by the left hand side of the identity (1). We put $V_1 = \{e_1, \cdots, e_p\}$ and $V_2 = \{e_{p+1}, \cdots, e_n\}$. First, we have

$$(2)\quad \begin{aligned} (X^{a_1} \wedge \cdots \wedge X^{a_p})(e_1 \wedge \cdots \wedge e_p) &= (1/p!) \sum_{\mathfrak{S}_p} (-1)^\sigma X^{a_1} e_{\sigma(1)} \wedge \cdots \wedge X^{a_p} e_{\sigma(p)} \\ &= (1/p!) \sum_{\mathfrak{S}_p} (-1)^\sigma \alpha_{\sigma(1)}^{a_1} \cdots \alpha_{\sigma(p)}^{a_p} e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(p)} \\ &= (1/p!) \sum_{\mathfrak{S}_p} \alpha_1^{a_{\sigma(1)}} \cdots \alpha_p^{a_{\sigma(p)}} e_1 \wedge \cdots \wedge e_p. \end{aligned}$$

We denote by S_λ and T_λ the Schur functions corresponding to the partition $\lambda = (\lambda_1, \cdots, \lambda_s)$ $(\lambda_1 \geq \cdots \geq \lambda_s > 0)$ with variables $\{\alpha_1, \cdots, \alpha_p\}$ and $\{\alpha_{p+1}, \cdots, \alpha_n\}$,