# 17. *The Vanishing Viscosity Method and a Two-phase Stefan Problem with Nonlinear Flux Condition of Signorini Type*

By Nobuyuki KENMOCHI*) and Irena PAWLOW**)

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1987)

**1. Introduction.** This paper is concerned with a two-phase Stefan problem with nonlinear flux condition of the so-called Signorini type. Let $\Omega$ be a bounded domain in $R^N$ $(N\geqq 2)$ whose boundary consists of two smooth disjoint surfaces $\Gamma_0$, $\Gamma_1$, and let $T$ be a fixed positive number, $Q=(0,T)\times\Omega$, $\Sigma_0=(0,T)\times\Gamma_0$, and $\Sigma_1=(0,T)\times\Gamma_1$. The problem, denoted by (P), is to find a function $u=u(t,x)$ on $Q$ satisfying

$$\begin{aligned}&u_t-\Delta\beta(u)=0 \qquad \text{in } Q,\\ &u(0,\cdot)=u_0 \qquad \text{in } \Omega,\\ &\beta(u)=g_0 \qquad \text{on } \Sigma_0,\\ &-\frac{\partial\beta(u)}{\partial n}\in\gamma(\beta(u)-g_1) \qquad \text{on } \Sigma_1.\end{aligned}$$

Here $\beta: R\to R$ is a given function which vanishes on $[0,1]$, is non-decreasing on $R$ and bi-Lipschitz continuous both on $(-\infty,0]$ and $[1,+\infty)$; $\gamma$ is a multivalued function from $R$ into $R$ given by $\gamma(r)=0$ for $r>0$, $\gamma(0)=(-\infty,0]$ and $\gamma(r)=\emptyset$ for $r<0$; $u_0$ is a given initial datum; $g_0$ and $g_1$ are given functions on $\Sigma_0$ and $\Sigma_1$, respectively; $(\partial/\partial n)$ denotes the outward normal derivative. For the data we postulate that

(A1) $g_i$ $(i=0,1)$ is the trace of a function, denoted by $g_i$ again, on $Q$ such that $g_i\in W^{1,2}(0,T;H^1(\Omega))\cap L^\infty(0,T;H^2(\Omega))$, $m_0\leqq g_0\leqq m_0'$, $m_1\geqq g_1\geqq m_1'$ a.e. on $Q$, where $m_0\leqq m_0'<0$, $m_1\geqq m_1'>0$ are constants.

(A2) (i) $u_0\in L^\infty(\Omega)$, meas. $\{x\in\Omega;0\leqq u_0(x)\leqq 1\}=0$, $v_0=\beta(u_0)\in H^1(\Omega)$; (ii) $v_0=g_0(0,\cdot)$ a.e. on $\Gamma_0$, $v_0\geqq g_1(0,\cdot)$ a.e. on $\Gamma_1$; (iii) there are constants $\delta>0$, $k_0<0$, $k_1>0$ such that $v_0\leqq k_0$ a.e. on $\Omega_{0,\delta}$ and $v_0\geqq k_1$ a.e. on $\Omega_{1,\delta}$, where

$$\Omega_{i,\delta}=\{x\in\Omega;\ \mathrm{dist.}(x,\Gamma_i)<\delta\},\qquad i=0,1.$$

In particular, when $g_0$ and $g_1$ are independent of time $t$, problem (P) was treated by Magenes-Verdi-Visintin [6] in the framework of nonlinear contraction semigroups in $L^1(\Omega)$ (cf. Bénilan [1], Crandall [3]), and the solution is unique in the sense of Crandall-Liggett [4]. Also, in case the flux condition is of the form $-(\partial/\partial n)\beta(u)=\gamma(t,x,\beta(u))$, with smooth function $\gamma(t,x,r)$ on $\Sigma_1\times R$, the problem was uniquely solved in variational sense by Niezgodka-Pawlow [7], Visintin [9] and Niezgodka-Pawlow-Visintin [8].

*) Department of Mathematics, Faculty of Education, Chiba University, Chiba, Japan.

**) Polish Academy of Sciences, Systems Research Institute, Newelska 6, 01-447 Warsaw, Poland.