15. *A Formulation of Noncommutative McMillan Theorem*

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§1. Introduction. In this short note, we formulate and prove a McMillan type convergence theorem in a noncommutative dynamical system based on our works about entropy operators [1].

Before formulating the McMillan theorem, we discuss a description of noncommutative dynamical systems, a noncommutative message space and entropy operators.

A noncommutative dynamical system (NDS for short) can be described by a von Neumann algebraic triple or, more generally, a C^*-algebraic triple denoted by $(\mathfrak{N}, \mathfrak{S}, \alpha)$. Namely, $\mathfrak{N}$ is a von Neumann algebra or C^*-algebra, $\mathfrak{S}$ is the set of all states on $\mathfrak{N}$ and α is an automorphism of $\mathfrak{N}$ describing a certain evolution of the system. A self-adjoint element A of the algebra $\mathfrak{N}$ corresponds to a random variable in usual commutative dynamical (probability) systems (CDS for short) and a state in NDS corresponds to a probability measure in CDS. Here we use a von Neumann algebraic description for simplicity. Consult the bibliography [2] for NDS and noncommutative probability theory.

Let $\mathfrak{N}$ be a finite dimensional von Neumann (matrix) algebra acting on a Hilbert space $\mathcal{H}$ with a faithful normal tracial state τ, and let $P(\mathfrak{M})$ be the set of all *minimal finite partitions of unit I* in a von Neumann subalgebra $\mathfrak{M}$ of $\mathfrak{N}$. A set of projections $\tilde{P}=\{P_j\}$ is said to be a minimal partition of I in $\mathfrak{M}$ if $P_j \in \mathfrak{M}(\forall j)$, $P_i \perp P_j$ $(i \neq j)$ and $\sum P_j = I$ hold, and for each j there does not exist a projection E such as $0<E<P_j$. Since any two partitions $\tilde{P}=\{P_j\}$ and $\tilde{Q}=\{Q_j\}$ are unitary equivalent, *the entropy operator* $H_\tau(\mathfrak{M})$ and *the entropy* $S_\tau(\mathfrak{M})$ w.r.t. $\mathfrak{M}$ and τ can be uniquely defined as [1]:

$$H_\tau(\mathfrak{M}) = -\sum_k P_k \log \tau(P_k) \tag{1.1}$$

$$S_\tau(\mathfrak{M}) = \tau(H_\tau(\mathfrak{M})) \tag{1.2}$$

for any $\tilde{P}=\{P_j\} \in P(\mathfrak{M})$. The above entropy $S_\tau(\mathfrak{M})$ has already been discussed in [3, 4] without considering $H_\tau(\mathfrak{M})$.

Now for any von Neumann subalgebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{N}$ and any partition $\tilde{P}=\{P_j\} \in P(\mathfrak{M}_2)$, it is easily seen that $\tilde{P}$ is not always in $P(\mathfrak{M}_1 \vee \mathfrak{M}_2)$ but there exists a partition $\{P_{ij}\}$ in $P(\mathfrak{M}_1 \vee \mathfrak{M}_2)$ such that $P_j = \sum_i P_{ij}$, where

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