2. $A \geqq B \geqq 0$ *iff* $(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$ *for* $r \geqq 0, p \geqq 0, q \geqq 1$ *with* $(1+2r)q \geqq p+2r$

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A capital letter means a bounded linear opeartor on a Hilbert space. An operator T is said to be positive in case $(Tx, x) \geqq 0$ for every x in a Hilbert space.

What functions preserve the ordering of positive operators? In other words, what must f satisfy so that

$$A \geqq B \geqq 0 \quad \text{implies} \quad f(A) \geqq f(B)?$$

A function f is said to be operator monotone if f satisfies the property stated above. This problem was first studied by K. Löwner, who had given a complete description of operator monotone functions. Also he had shown the following result in [7].

Theorem A. *If A and B are bounded positive operators on a Hilbert space such that $A \geqq B \geqq 0$, then $A^\alpha \geqq B^\alpha$ for each α in the interval* $[0, 1]$.

This theorem had been also shown by E. Heinz [4] and also T. Kato [5] had given a shorter proof. Recently two simple proofs have been shown by Au-Yeung [1] and Man Kam Kwong [6]. An elegant and simple proof based on C^*-algebra theory of Theorem A has been shown in [8].

Nevertheless it is well known that $A \geqq B \geqq 0$ does not always assure $A^2 \geqq B^2$ in general. We know almost no knowledge except both commutative case and operator monotone function case.

The purpose of this paper is to announce early "order preserving inequalities" on A and B in case $A \geqq B \geqq 0$, that is, we have found two order preserving functions $f(X)$ and $g(Y)$ under suitable and agreeable additional conditions. We explain these functions in Remark 1 and also these conditions in Remark 3.

Theorem 1. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

(i) $$(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$$

and

(ii) $$A^{(p+2r)/q} \geqq (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geqq 0$, $q \geqq 1$ and $(1+2r)q \geqq p+2r$.

Corollary 1. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

(i) $$(B^r A^p B^r)^{(1+2r)/(p+2r)} \geqq B^{1+2r}$$

and

(ii) $$A^{1+2r} \geqq (A^r B^p A^r)^{(1+2r)/(p+2r)}$$

hold for each $p \geqq 1$.

Corollary 2. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*