112. Gram's Law for the Zeta Zeros and the Eigenvalues of Gaussian Unitary Ensembles

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§1. Let γ run over the positive imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. Let $\Gamma(s)$ be the Γ-function and $\vartheta(t)=\mathrm{Im}(\log\Gamma(1/4+(1/2)it))-(1/2)t\log\pi$, where t is a real number. We define g_x by $\vartheta(g_x)=x\pi$ for $x\geqq -1$, where $\vartheta(t)$ is strictly increasing for $t\geqq 7$. Here we are concerned with the following problem.

Problem. *To study the quantity defined by*

$$\lim_{M\to\infty}\frac{1}{M}G_M(k,\alpha)$$

for each integer $k\geqq 0$ and any positive number α, where we put

$$G_M(k,\alpha)=\left|\left\{-1\leqq m\leqq M\,;\left|\left\{\gamma\leqq\vartheta^{-1}(\pi(M+1))\,;\,\frac{1}{\pi}\vartheta(\gamma)\in[m,m+\alpha)\right\}\right|=k\right\}\right|,$$

γ being counted with multiplicity.

We recall two observations concerning this problem. First Gram observed more than eighty years ago that the zeros of $\zeta(1/2+it)$ appears exactly once in the interval (g_m, g_{m+1}) up to $t\leqq 50$. This phenomenon, which seemed to Gram to continue also for $t>50$, has been called Gram's law although many counter-examples have been observed since Hutchinson (cf. chapters 6, 7 and 8 of [2] for a detailed description of the history). Gram's law implies that for any integer $M\geqq -1$, $G_M(k,1)=M+2$ if $k=1$ and $=0$ if $k\neq 1$. Second, the latest computer calculations by van de Lune, te Riele and Winter [14] tell us that for $M=1500000000$, $(1/M)G_M(0,1)=0.1378\cdots$, $(1/M)G_M(1,1)=0.7261\cdots$, $(1/M)G_M(2,1)=0.1342\cdots$ and $(1/M)G_M(3.1)=0.0018\cdots$ and that $(1/M)G_M(k,1)$ increases for $k=0$, 2 and 3 and decreases for $k=1$ as M becomes larger. We remark here that in both observations, all the non-trivial zeros of $\zeta(s)$ are on the critical line and are simple as far as they have calculated. In this note we shall state some results and conjectures concerning the above problem.

§2. We denote the number of the non-trivial zeros of $\zeta(s)$ in $0<\mathrm{Im}(s)<t$ by $N(t)$ as usual. Since

$$G_M(k,\alpha)=|\{-1\leqq m\leqq M\,;\,N(g_{m+\alpha})-N(g_m)=k\}|$$

and $N(t)=\pi^{-1}\vartheta(t)+1+S(t)$ for $t\geqq t_0$, the following means

$$\sum_{m\leqq M}(S(g_{m+\alpha})-S(g_m))^j \qquad \text{for any integer } j\geqq 1$$

must give some information on our problem, where we put $S(t)=(1/\pi)arg\,\zeta(1/2+it)$ as usual. The above sum is a discrete version of the integral