108. *Degenerate Self-Adjoint Perturbation in Hilbert Space*

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§1. Introduction. Among many perturbation operators appearing in differential equations, self-adjoint perturbations constitute a special class because of their nice properties. The purpose of this paper is to develop a theory of a self-adjoint perturbation added to an unbounded self-adjoint operator in a Hilbert space. The perturbation in this paper is a degenerate or a finite-dimensional one which has a physical interpretation as a feedback in control systems. The perturbed operator has a positive parameter. It is studied how the minimum eigenvalue of it behaves as the parameter increases.

We begin with the formulation of the problem. Let H be a real Hilbert space with an inner product and a norm which are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively. Throughout the paper, L will denote an unbounded self-adjoint linear operator with domain $\mathcal{D}(L)$ dense in H. It is assumed that L is positive definite and has compact resolvent. As is well known [2], there is a set of eigenpairs $\{\lambda_i, \phi_{ij}\}$ for L satisfying the following conditions:

(i) $\sigma(L)=\{\lambda_1, \lambda_2, \cdots\}$; $0<\lambda_1<\lambda_2<\cdots<\lambda_i<\cdots\to\infty$;

(ii) $L\phi_{ij}=\lambda_i\phi_{ij}$, $i\geq 1$, $1\leq j\leq m_i(<\infty)$; and

(iii) the set $\{\phi_{ij}\,;\, i\geq 1, 1\leq j\leq m_i\}$ forms a complete orthonormal system in H.

Given a set $\{\psi_1, \cdots, \psi_N\}\subset H$, let us define an operator B as

$$Bx=\sum_{i=1}^{N}\langle x, \psi_i\rangle\psi_i, \qquad x\in H.$$

It is clear that B is self adjoint and nonnegative. Elements ψ_i's are physically interpreted as sensors and actuators in feedback control systems. The operator B is added to L, and the perturbed operator then becomes

$$L+kB=L+k\sum_{i=1}^{N}\langle \cdot, \psi_i\rangle\psi_i, \tag{1}$$

where k indicates a positive parameter. Since B is bounded, $L+kB$ is also a positive-definite self-adjoint operator with domain $\mathcal{D}(L+kB)=\mathcal{D}(L)$, and has compact resolvent. The minimum eigenvalue of $L+kB$ is denoted by $\mu(k)$, and will play an important role since it determines the decay rate of the semigroup $e^{-t(L+kB)}$ generated by the differential equation in H;

$$\frac{dx}{dt}=-(L+kB)x, \quad t>0, \quad x(0)=x_0.$$

It is easy to derive that

$$\begin{aligned}\mu(k)&=\inf_{x\in\mathcal{D}(L),\|x\|=1}\langle (L+kB)x, x\rangle \\ &=\inf_{x\in\mathcal{D}(L^{1/2}),\|x\|=1}\{\|L^{1/2}x\|^2+k\langle Bx, x\rangle\}.\end{aligned}\tag{2}$$