106. *Uniform Distribution of the Zeros of the Riemann Zeta Function and the Mean Value Theorems of Dirichlet L-functions*

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(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1987)

We shall give a brief survey of some applications of our previous works on the uniform distribution of the zeros of the Riemann zeta function $\zeta(s)$ (cf. [1], [2]). The details will appear elsewhere. We assume the Riemann Hypothesis throughout this article.

Let γ run over the positive imaginary parts of the zeros of $\zeta(s)$. We may recall the following two theorems which are special cases of the more general theorem in the author's [2]. The first theorem is a refinement of Landau's theorem (cf. [5]). We put $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geqq 1$ and $\Lambda(x)=0$ otherwise.

Theorem 1. *For any positive* α,

$$\sum_{0<\gamma\leqq T} e^{i\alpha\gamma}=-\frac{1}{2\pi}\frac{\Lambda(e^\alpha)}{e^{\alpha/2}}T+\frac{e^{i\alpha T}}{2\pi i\alpha}\log T+O\left(\frac{\log T}{\log\log T}\right).$$

The second theorem gives us a connection of the distribution of γ with a rational number.

Theorem 2. *For any positive* α,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{0<\gamma\leqq T} e^{i\gamma\log(\gamma/2\pi e\alpha)}=\begin{cases}-e^{(1/4)\pi i}\dfrac{C(\alpha)}{2\pi} & \text{if } \alpha \text{ is rational}\\ 0 & \text{if } \alpha \text{ is irrational,}\end{cases}$$

where we put $C(\alpha)=\dfrac{1}{\sqrt{\alpha}}\dfrac{\mu(q)}{\varphi(q)}$ *with the Möbius function* $\mu(q)$ *and the Euler function* $\varphi(q)$ *if* $\alpha=a/q$ *with relatively prime integers* a *and* $q\geqq 1$.

We should remark that the remainder terms in Theorems 1 and 2 depend on α heavily. In our applications with which we are concerned here it is necessary and important to clarify the dependences on α. In fact, if we follow the proofs of our theorems above in pp. 103-112 of [2], then we get the following explicit versions of them.

Theorem 1′. *Let* $0<Y_0<Y\leqq T$. *Then*

$$\begin{aligned}\sum_{Y_0<\gamma\leqq Y} e^{i\alpha\gamma}&=A(\alpha,Y,Y_0)+O((\alpha e^{(1/2)\alpha}+1)\log Y/\log\log Y)\\ &\quad-\frac{\alpha}{2\pi}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^{1+\delta}\log k}e^{(1/2+\delta)\alpha}\int_{Y_0}^{Y}e^{-it\log k+it\alpha}\,dt\\ &\quad-\frac{\alpha}{2\pi}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^{1+\delta}\log k}e^{-(1/2+\delta)\alpha}\int_{Y_0}^{Y}e^{it\log k+it\alpha}\,dt\end{aligned}$$

uniformly for a positive α, *where we put* $\delta=1/\log T$ *and*