11. *On a Problem of Yamamoto Concerning Biquadratic Gauss Sums*

By Hiroshi ITO
Department of Mathematics, Faculty of Science,
Nagoya University

(Communicated by Shokichi IYANAGA, M. J. A., Feb. 12, 1987)

Yamamoto [4] observed some relations, which can be stated as (I) $\sim$ (IV) below, between the biquadratic Gauss sums and the generalized Bernoulli numbers defined for prime numbers $p<4,000$ such that $p\equiv 1 \pmod 4$. He proposed the question whether these relations were always true. In this note, we report counter-examples to (I) $\sim$ (III) and prove (IV). Some remarks on a problem which still remains will be added.

1. For a prime number $p\equiv 1 \pmod 4$, take positive integers a and b such that $p=a^2+4b^2$ and put $\omega=\omega_p:=a+2bi$. Define the Dirichlet character $\chi=\chi_p$ modulo p by $\chi(m)=(m/\omega)_4$ where $(m/\omega)_4$ is the biquadratic residue symbol in $\boldsymbol{Q}(i)$, the Gauss' number field. Consider the Gauss sum

$$\tau(\chi)=\sum_{m=1}^{p-1}\chi(m)e^{2\pi i m/p}.$$

We write

$$\tau(\chi)=\varepsilon_p\omega^{1/2}p^{1/4} \quad \text{with} \quad 0<\arg(\omega^{1/2})<\pi/4.$$

It is a classical result that $\varepsilon_p^4=1$. On the other hand we put

$$A_p=-\frac{1}{p}\sum_{m=1}^{p-1}\chi(m)m, \quad B_p=-\frac{1}{p}\sum_{m=1}^{(p-1)/2}\chi(m)m, \quad C_p=\sum_{m=1}^{(p-1)/2}\chi(m).$$

Then the assertions (I) $\sim$ (IV) in [4] (p. 212) can be stated as follows:

(I) $\quad -\pi/4\leq\arg(\varepsilon_p\bar{A}_p)<3\pi/4 \quad$ if $p\equiv 5 \pmod 8$,

(II) $\quad -\pi\leq\arg(\varepsilon_p\bar{B}_p)<0 \quad$ if $p\equiv 5 \pmod 8$,

(III) $\quad \mathrm{Im}(\varepsilon_p\bar{C}_p)>0 \quad$ if $p\equiv 5 \pmod 8$,

(IV) $\quad \mathrm{Re}(\varepsilon_p\bar{B}_p)>0 \quad$ if $p\equiv 1 \pmod 8$,

where the bar indicates the complex conjugation. Note that $A_p=C_p=0$ if $p\equiv 1 \pmod 8$, when the character χ satisfies $\chi(-1)=1$.

At the time when [4] was published, the calculation of ε_p was very hard. Now we have an elegant expression of ε_p obtained by Matthews [3]. Define $\delta_p\in\{1,-1\}$ by

$$\{(p-1)/2\}!\equiv\delta_p i \pmod{\omega}.$$

Then it follows from [3] that

$$\varepsilon_p=-\delta_p\chi(2i)\left(\frac{b}{a}\right)_2\times\begin{cases} i & \text{if } a\equiv 1 \pmod 4,\\ 1 & \text{if } a\equiv 3 \pmod 4,\end{cases}$$

where $(b/a)_2$ is the (rational) Jacobi symbol. By means of this expression for ε_p, the author examined, with the help of an electronic computer, the assertions (I) $\sim$ (III) for $p<1,000,000$ and found counter-examples. There