94. Necessary and Sufficient Conditions for the Convergence of Formal Solutions

By Masafumi YOSHINO
Department of Mathematics, Tokyo Metropolitan University

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§1. **Introduction.** In this paper we shall give a necessary and sufficient condition for the convergence of formal solutions of certain type of analytic equations of general independent variables. The result here is an extension of that of [3] to equations of general independent variables, which coincides with the result of [3] in the case of two independent variables.

§2. **Statement of results.** Let $x=(x_1, \cdots, x_d)$ $(d\geq 2)$ be the variable in $\boldsymbol{C}^d$. For $\eta\in\boldsymbol{R}^d$ and a multi-index $\alpha=(\alpha_1, \cdots, \alpha_d)\in\boldsymbol{N}^d$, $N=\{0,1,2,\cdots\}$, we set $\eta^\alpha=\eta_1^{\alpha_1}\cdots\eta_d^{\alpha_d}$ and $(x\cdot\partial)^\alpha=(x_1\partial_1)^{\alpha_1}\cdots(x_d\partial_d)^{\alpha_d}$, where $\partial=(\partial_1, \cdots, \partial_d)$ and $\partial_j=\partial/\partial x_j$ $(j=1, \cdots, d)$. Let $m\geq 1$ be an integer and let $\omega\in\boldsymbol{C}^d$. Then we are concerned with the convergence of all formal solutions of the form $u(x)=x^\omega\sum_{\eta\in\boldsymbol{N}^d}u_\eta x^\eta/\eta!$ of the equation

$$P(x\,;\partial)u\equiv\sum_{|\alpha|\leq m}a_\alpha(x)\partial^\alpha u(x)=f(x)x^\omega \tag{2.1}$$

where $a_\alpha(x)$ is analytic at the origin and $f(x)$ is a given analytic function. We say that a formal solution $u=x^\omega\sum_{\eta\in\boldsymbol{N}^d}u_\eta x^\eta/\eta!$ converges if the sum $\sum_{\eta\in\boldsymbol{N}^d}u_\eta x^\eta/\eta!$ converges and represents an analytic function in x. Let us expand $a_\alpha(x)$ into the power of x, $a_\alpha(x)=\sum_\gamma a_{\alpha,\gamma}x^\gamma/\gamma!$, and let us define

$$M_P=\{\gamma-\alpha\in\boldsymbol{Z}^d\,;\,a_{\alpha,\gamma}\neq 0 \text{ for some } \alpha \text{ and } \gamma\}. \tag{2.2}$$

Then we assume

(A.1) $M_P\subset\{\eta\in\boldsymbol{R}^d\,;\,\eta_1+\cdots+\eta_d\geq 0\}$ and $M_P\cap\{\eta\in\boldsymbol{R}^d\,;\,\eta_1+\cdots+\eta_d=0\}$ is contained in some proper cone with apex at the origin.

We define the set Γ_0 by $\Gamma_0=$Convex hull of $\{t\theta\in\boldsymbol{R}^d\,;\,t\geq 0,\ \theta\in M_P\}$.

We set $p(\eta)=\sum_{|\alpha|\leq m}a_{\alpha,\alpha}x^\alpha\partial^\alpha/\alpha!$, and we denote by $p_m(\eta)$ the m-th homogeneous part of $p(\eta)$. For $\xi\in\boldsymbol{R}^d$, $|\xi|=1$, we set $\Gamma(\xi\,;\varepsilon)=\{\eta\in\boldsymbol{R}^d\,;\,|\eta/|\eta|-\xi|<\varepsilon\}$. Then we define the quantity $\sigma_{\xi,\varepsilon}$ by

$$\sigma_{\xi,\varepsilon}=\sup\{c\in\boldsymbol{R}\,;\ \liminf_{|\eta|\to\infty,\eta\in\Gamma(\xi,\varepsilon)\cap\boldsymbol{Z}^d}|\eta|^{-c}|p(\eta+\omega)|>0\}, \tag{2.3}$$

where if $\liminf|\eta|^{-c}|p(\eta+\omega)|=0$ for every $c\in\boldsymbol{R}$, we put $\sigma_{\xi,\varepsilon}=-\infty$. Note that $\sigma_{\xi,\varepsilon}\leq m$, since $p(\eta+\omega)$ is of degree m. Since $\sigma_{\xi,\varepsilon}$ increases as ε tends to zero, we set $\sigma_\xi\equiv\lim_{\varepsilon\downarrow 0}\sigma_{\xi,\varepsilon}$. For the fundamental property of σ_ξ we refer to [3].

We define a differential operator $Q(x\,;\partial)\equiv\sum_{|\beta|\leq m_0}b_\beta(x)\partial^\beta$ by

$$Q(x\,;\partial)=P(x,\partial)-\sum_{|\alpha|\leq m}a_{\alpha,\alpha}x^\alpha/\alpha!,$$

where $m_0\leq m$.

Let us take θ, $|\theta|=1$ such that $p_m(\theta)\neq 0$. We write $\eta=\zeta_1\theta+\zeta'$, $\zeta'=$