80. *A Generalization of Itô's Lemma*

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1. Introduction. In the traditional definition of K. Itô's stochastic integral of a process φ with respect to Brownian motion B it is essential that φ be non-anticipatory [8]. However, there are some works in which one has tried to avoid this condition, s. e.g. [1, 4, 9]. Finally, the white noise analysis, advocated by T. Hida (e.g. [2, 3]), has provided a framework, in which stochastic integrals can be naturally defined without posing such measurability conditions, as has been shown in a recent paper by H.-H. Kuo and A. Russek [7].

Let $(\mathcal{S}'(\boldsymbol{R}), \mathcal{B}, d\mu)$ be white noise, i.e. $\mathcal{B}$ is the σ-algebra over $\mathcal{S}'(\boldsymbol{R})$ generated by the cylinder sets and μ is the Gaussian measure on $\mathcal{B}$ with characteristic functional

$$\exp(-1/2\,\|\xi\|_2^2)=\int_{\mathcal{S}'(\boldsymbol{R})}\exp(i\langle x,\xi\rangle)d\mu(x) \tag{1.1}$$

for $\xi\in\mathcal{S}(\boldsymbol{R})$, $\|\cdot\|_2$ denoting the norm of $L^2(\boldsymbol{R}, dt)$ and $\langle\cdot,\cdot\rangle$ the canonical duality. By (L^p), $p>0$, we denote the Banach space $L^p(\mathcal{S}'(\boldsymbol{R}), \mathcal{B}, d\mu)$. Note that

$$B(t\,;x):=\langle x, 1_{(0,t)}\rangle, \qquad x\in\mathcal{S}'(\boldsymbol{R}) \tag{1.2}$$

(although not pointwise defined) is a well-defined random variable in (L^p), $p\geq 1$, and a Brownian motion (under $d\mu$).

In [2, 3] Hida introduced the space $(L^2)^+$ of testfunctionals of white noise and its dual $(L^2)^-$ of generalized functionals. Furthermore he defined the operators ∂_t, $t\in\boldsymbol{R}$, which are partial derivatives $\partial/\partial x(t)$ for white noise testfunctionals, cf. also [5, 6]. Since ∂_t is densely defined on $(L^2)^+$ there is its adjoint ∂_t^* acting on $(L^2)^-$. Note that we have the Gel'fand triple

$$(L^2)^-\supset(L^2)\supset(L^2)^+ \tag{1.3}$$

so that ∂_t^* acts by restriction on (L^2).

The following was shown in the paper [7] of Kuo and Russek: assume that φ is a map from $\boldsymbol{R}_+$ into (L^2), non-anticipatory (i.e. for each $t\in\boldsymbol{R}_+$, $\varphi(t)$ is measurable w.r.t. $\sigma(B(s\,;\cdot), 0\leq s\leq t)$) and

$$\int_a^b \boldsymbol{E}(|\varphi(t)|^2)dt \tag{1.4}$$

is finite, then

$$\int_a^b \partial_t^*\varphi(t)dt \tag{1.5}$$

exists in (L^2) and equals Itô's stochastic integral of φ w.r.t. Brownian motion. Of course, this generalizes to higher-dimensional Brownian