76. A Note on p-adic Etale Cohomology

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1. Let X be a projective smooth scheme over a complete discrete valuation ring A of mixed characteristics $(0,p)$. In [2], Fontaine and Messing studied the relation between the p-adic etale cohomology of the generic fiber $H^*_{et}(X_{\bar\eta})=H^*_{et}(X_\eta\otimes\bar\eta, \boldsymbol{Z}_p)$ ($\bar\eta$ is an algebraic closure of η) and the crystalline cohomology of the special fiber $H^*_{crys}(X_s)$. In this article, we consider not $Gal(\bar\eta/\eta)$-representation $H^*_{et}(X_{\bar\eta})$, but $H^*_{et}(X_\eta)$ itself and study this cohomology group by using the syntomic cohomology introduced in [2]. Detailed studies containing the complete proof will appear elsewhere.

We will use the following notation: X is a projective, smooth and geometrically connected scheme over A of dimension d as above, and $Y=X_s$ (resp. X_η) is the special fiber (resp. the generic fiber), and $i: Y\to X$ (resp. $j: X_\eta\to X$) is the canonical morphism. We assume that the residue field F of A has a finite p-base of order g (i.e. $[F:F^p]=p^g$).

Fontaine and Messing [2] defined the syntomic site X_{syn} and a sheaf S^r_n on X_{syn} in order to link the etale cohomology to De Rham cohomology. This sheaf S^r_n is regarded as an "ideal" etale sheaf $\boldsymbol{Z}/p^n(r)$ on X. Namely, the group $H^q(X_{syn}, S^r_n)$ is expected to play a role of "$H^q(X_{et}, \boldsymbol{Z}/p^n(r))$" which cannot be defined directly. In [2], a global cohomology $H^q(X_{\bar\eta}, \boldsymbol{Z}_p)$ was studied under the assumption $e_A=\mathrm{ord}_A(p)=1$. Our aim in this paper is a local study of p-adic etale vanishing cycles $i^*Rj_*\boldsymbol{Z}/p^n(r)$ when e_A may not be 1. Put $\mathcal{S}_n(r)=i^*R\pi_*S^r_n\in D(Y_{et})$ as in [3] where $\pi: X_{syn}\to X_{et}$ is the canonical morphism. Fontaine and Messing defined a morphism $S^r_n\to i'^*j'_*\boldsymbol{Z}/p^n(r)$ (where $j': X_{\eta et}\to X_{syn-et}$, $i': X_{syn}\to X_{syn-et}$) in [2] 5, which induces $\mathcal{S}_n(r)\to i^*Rj_*\boldsymbol{Z}/p^n(r)$. We study the difference between $\mathcal{S}_n(r)$ and $i^*Rj_*\boldsymbol{Z}/p^n(r)$.

Theorem. *If $r<p-1$, there exists a distinguished triangle*

$$\mathcal{S}_n(r)\longrightarrow\tau_{\le r}i^*Rj_*\boldsymbol{Z}/p^n(r)\longrightarrow W_n\Omega^{r-1}_{Y\,lo}\,[-r].$$

where $W_n\Omega^{r-1}_{Y\,log}$ is the logarithmic Hodge-Witt sheaf. In particular, if $r\ge d(=dim\,X)+g(=ord_p\,[F:F^p])$, we have a long exact sequence

$$\longrightarrow H^q(X_{syn}, S^r_n)\longrightarrow H^q(X_{\eta\,et}, \boldsymbol{Z}/p^n(r))\longrightarrow H^{q-r}(Y_{et}, W_n\Omega^{r-1}_{Y\,log})\longrightarrow$$
$$H^{q+1}(X_{syn}, S^r_n)\longrightarrow H^{q+1}(X_{\eta\,et}, \boldsymbol{Z}/p^n(r))\longrightarrow H^{q-r+1}(Y_{et}, W_n\Omega^{r-1}_{Y\,log})\longrightarrow.$$

In the case $e_A=ord_A(p)=1$ and $r\ge d+g$, considering

$$\mathcal{S}_n(r)\simeq DR(X\otimes\boldsymbol{Z}/p^n)[-1]$$

($DR(T)$ means the De Rham complex $\Omega^\bullet_{T/\boldsymbol{Z}}$), we have

Corollary 1. *Suppose that $e_A=ord_A(p)=1$ and $d+g\le r<p-1$. Then,*