64. A Generalization of Lefschetz Theorem

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We improve the classical Lefschetz theorem as follows:

Theorem. *Let A be an effective ample divisor on an algebraic variety V defined over $\boldsymbol{C}$ of dimension n, let v be a point on $V-A$ such that $V-A-v$ is smooth and set $U=V-v$. Then the relative homotopy group $\pi_k(U, A)$ vanishes for every $k<n$.*

Using Morse theory, we prove this theorem by modifying Andreotti-Frankel method (cf. [1], [2]). First, replacing A by mA for $m\gg 1$ if necessary, we may assume that A is very ample. Thus $V\subset \boldsymbol{P}^N$ and $A=V\cap S$ for some hyperplane S in $\boldsymbol{P}^N$. We fix an affine linear coordinate of $\boldsymbol{P}^N-S\simeq \boldsymbol{C}^N$ and let δ denote the Euclid distance with respect to this coordinate. Set $N_R=\{x\in V-A \mid \delta(x, v)\leqq R\}$ and $U_R=V-N_R$ for each $R>0$. If $r>0$ is small enough, the function $d(x)=\delta(x, v)$ has no critical point in N_{4r}. Hence U_{3r} and U_r are deformation retracts of U.

For a point p in $\boldsymbol{P}^N-S-V$, let f be the function $\delta(x, p)^2$ on $U-A$. By [2; Theorem 6.6], f has no degenerate critical points for almost all p. In particular, we can choose p such that $\delta(p, v)<r$. Set $T_a=A\cup\{x\in V-A \mid f(x)\geqq a^2\}$. Then $T_L\subset U_{3r}\subset T_{2r}\subset U_r$ for any $L\gg 1$. Using Morse theory similarly as in [2; p. 42], we infer that T_{2r} has the homotopy type of T_L with finitely many cells of real dimension $\geqq n+1$ attached, so we obtain $\pi_k(T_{2r}, A)\simeq\pi_k(T_L, A)\simeq\{1\}$ for $k<n$. On the other hand, the composition $\pi_k(U_{3r}, A)\to\pi_k(T_{2r}, A)\to\pi_k(U_r, A)\simeq\pi_k(U, A)$ is bijective. Hence $\pi_k(U, A)$ is trivial. Thus we complete the proof.

Corollary. *Let L be the total space of an ample line bundle on a compact complex manifold M and let X be a compact analytic subspace of L of pure dimension $n=\dim M$. Then, for the natural map $f: X\to M$,*

1) *$\pi_k(f): \pi_k(X)\to\pi_k(M)$ is bijective if $k<n$ and is surjective if $k=n$.*

2) *$H_k(f): H_k(X; \boldsymbol{Z})\to H_k(M; \boldsymbol{Z})$ is bijective if $k<n$ and is surjective if $k=n$.*

3) *$H^k(f): H^k(M; \boldsymbol{Z})\to H^k(X; \boldsymbol{Z})$ is bijective if $k<n$ and is injective with torsion free cokernel if $k=n$.*

4) *$\mathrm{Pic}(M)\to\mathrm{Pic}(X)$ is bijective if $n>2$ and is injective if $n=2$. When $n=2$, the cokernel is torsion free if $H^1(M, \mathcal{O}_M)\to H^1(X, \mathcal{O}_X)$ is injective.*

Proof. Set $\mathcal{L}=\mathcal{O}_M[L]$, $\mathcal{S}=\mathcal{O}_M\oplus\mathcal{L}$, $P=\boldsymbol{P}(\mathcal{S})$ and $H=\mathcal{O}_P(1)$. Then P is a $\boldsymbol{P}^1$-bundle over M and there are disjoint sections M_∞ and M_0 corresponding to quotient bundles $\mathcal{O}_M$ and $\mathcal{L}$ of $\mathcal{S}$, respectively. The open set $P-M_\infty$ is naturally isomorphic to L and M_0 is identified with the 0-section. So we