61. On Two Conjectures on Real Quadratic Fields

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Recently we learned from a paper of H. Yokoi [2] that there are two conjectures (C_1), (C_2) concerning the class numbers of real quadratic fields.

(C_1): Let l be a square-free integer of the form $l=q^2+4$ $(q\in \boldsymbol{N})$. Then there exist just 6 quadratic fields $\boldsymbol{Q}(\sqrt{l})$ of class number one.

(C_2): Let l be a square-free integer of the form $l=4q^2+1$ $(q\in \boldsymbol{N})$. Then there exist just 6 quadratic fields $\boldsymbol{Q}(\sqrt{l})$ of class number one.

In this paper, we shall prove that at least one of the two conjectures is true and that there are at most 7 quadratic fields $\boldsymbol{Q}(\sqrt{l})$ of class number one for the other case. Our result will follow from two theorems which are independent of each other. Theorem 1 follows from Tatuzawa's lower bound for $L(1,\chi)$ [1], and Theorem 2 is obtained by the results of Yokoi [2] and by the help of a computer (Macsyma) in our Department.

In the sequel, l will always denote a square-free integer of the form $l=q^2+4$ or $l=4q^2+1$ $(q\in \boldsymbol{N})$. We shall denote by $h(l)$ the class number of the quadratic field $\boldsymbol{Q}(\sqrt{l})$.

Theorem 1. *There exists at most one $l\geqq e^{16}$ with $h(l)=1$.*

Proof. By Dirichlet's class number formula, we have

$$h(l)=\frac{\sqrt{l}}{2\log u}L(1,\chi_l),$$

where χ_l is the Kronecker character belonging to the quadratic field $\boldsymbol{Q}(\sqrt{l})$ and u is the fundamental unit of $\boldsymbol{Q}(\sqrt{l})$. By the choice of l, we have

$$u=\begin{cases}(q+\sqrt{l})/2 & \text{if } l=q^2+4,\\ 2q+\sqrt{l} & \text{if } l=4q^2+1.\end{cases}$$

Assume that $l\geqq e^{16}$. By Theorem 2 of [1], we have

$$L(1,\chi_l)>\frac{1}{16}(0.655)l^{-(1/16)}$$

with one possible exception of l.[1)]

Case 1. $l=q^2+4$. Then $u=(q+\sqrt{l})/2<\sqrt{l}$ and

$$h(l)=\frac{\sqrt{l}}{2\log u}L(1,\chi_l)>\frac{\sqrt{l}}{2\log\sqrt{l}}\,\frac{1}{16}(0.655)l^{-(1/16)}=\frac{1}{16}(0.655)\frac{l^{7/16}}{\log l}.$$

Since $f(x)=x^{7/16}/\log x$ is increasing on $[e^{16},\infty)$, we have

$$h(l)>\frac{1}{16}(0.655)\frac{e^7}{16}=2.805\cdots>2.$$

Case 2. $l=4q^2+1$. Then $u=2q+\sqrt{l}<2\sqrt{l}$ and

1) Put $k=l$ and $\varepsilon=1/16$ in Theorem 2 of [1].