57. A Note on Modules

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Introduction. Let R be a fixed (not necessarily commutative) ring. Throughout this note, we are concerned with left R-modules $M, A, H, \cdots$. Like in Goldie [1], we shall use the following terminology. A non-zero submodule K of M is called *essential* in M (or M is an *essential extension* of K) if $K \cap A = 0$ for any other submodule A of M, implies $A = 0$. M has *finite Goldie dimension* (abbr. FGD) if M does not contain a direct sum of infinite number of non-zero submodules. Equivalently, M has finite Goldie dimension if for any strictly increasing sequence $H_0, H_1, \cdots$ of submodules of M, there is an integer i such that for every $k \geqslant i$, H_k is essential submodule in H_{k+1}. M is *uniform*, if every non-zero submodule of M is essential in M. Then it is proved (Goldie [1]) that in any module M with FGD, there exist non-zero uniform submodules $U_1, U_2, \cdots, U_n$ whose sum is direct and essential in M. The number n is independent of the uniform submodules. This number n is called the *Goldie dimension* of M and denoted by $\dim M$. It is easily proved that if M has FGD then every submodule of M has also FGD and $\dim K \leq \dim M$ (K being a submodule of M).

Furthermore, if K, A are submodules of M, and K is a maximal submodule of M such that $K \cap A = 0$, then we say that K is a *complement* of A (or a complement in M). It is easily proved that if K is a complement in M, if and only if there exists a submodule A in M such that $A \cap K = 0$ and $K' \cap A \neq 0$ for any submodule K' of M containing K. In this case we have $K + A$ is essential in M.

We are now introducing a notion "*E-irreducible submodule of M*". A submodule H of M is said to be E-irreducible if $H = K \cap J$, K and J are submodules of M, and H is essential in K, imply $H = K$ or $H = J$. Every complement submodule is an E-irreducible submodule, but the converse is not true.

Example 1. Consider Z, the ring of integers and Z_{12}, the ring of integers modulo 12. Write $R = Z$ and $M = Z_{12}$. Now the principal submodule K of M generated by 2, is E-irreducible submodule, but not a complement submodule.

Example 2. Consider $R = Z$ and $M = Z_8 \times Z_3$. Now the submodule $K = (4) \times (0)$ of M is not E-irreducible (since $K = (Z_8 \times (0)) \cap ((4) \times Z_3)$ and K is essential in $Z_8 \times (0)$).

The purpose of this note is to prove the following result.