54. Completely Integrable Symplectic Mapping

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1. Introduction. The complete integrability, whose concept was first introduced by Liouville, is still an important subject in the recent development of Hamiltonian systems [1], [8]. This concept exists not only in Hamiltonian mechanical systems but also in certain systems of evolution equations. Also, the complete integrability can be found in symplectic mappings, which is the theme treated in this paper. Though any exponential map of a Hamiltonian vector field is a symplectic mapping, the converse is not true. Therefore, it is not always the case that a symplectic mapping possesses properties similar to those of a Hamiltonian flow. As far as complete integrability, however, similar features can be seen.

Any Hamiltonian version of a discrete-time mechanics is expressed in terms of a symplectic mapping [5], and the mechanics has also a Lagrangian formulation characterized by the discrete variational principle [4], [6]. Then, discrete systems expressed as the Euler equations, which appear in various branches, are transformed to symplectic mappings.

The contents of this paper are arranged as follows. A definition of a completely integrable symplectic mapping is given, and a discrete version of Liouville's theorem is proved. Furthermore, it is shown that its behavior is ergodic on (a submanifold of) a torus under some conditions.

2. Complete integrable symplectic map and its properties. In this section, we introduce the concept of complete integrable discrete mechanical systems and derive two important properties.

Let (M, ω) be a $2N$-dimensional symplectic manifold, ω being a symplectic structure. We think of a symplectic mapping ϕ on M. This is a Hamiltonian version of a discrete mechanical system. In addition to preservation of the symplectic structure, ϕ has a formal similarity to the usual canonical equations. That is, when ϕ is sufficiently near the identity mapping, it is expressed on any symplectic chart (Q^i, P_j) as

$$Q^i_{\tau+1}-Q^i_{\tau}=\frac{\partial H}{\partial P_i}(Q_\tau, P_{\tau+1}), \tag{1a}$$

$$P_{i,\tau+1}-P_{i,\tau}=-\frac{\partial H}{\partial Q^i}(Q_\tau, P_{\tau+1}), \tag{1b}$$

where τ takes the integral values representing the discrete time and H is a certain function [5].

A smooth function f on M is called an F.I. of ϕ, if and only if $\phi^* f=f$ holds, where ϕ^* denotes the pull-back associated with ϕ. The value of