50. *Limiting Behaviour of Linear Cellular Automata*

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(Communicated by Kôsaku YOSIDA, M. J. A., June 9, 1987)

1. Introduction. Stephan J. Willson pointed out that cellular automata can generate fractals [1]. Specifically he showed that in any 2-state linear cellular automaton a sequence of space time pattern at time 2^n contracted by a factor $1/2^n$ converges to a limit when n tends to infinity. This result, however, can be generalized, and we will show in this paper that every linear cellular automaton has its "limit set". (Limit set is defined in section 2.)

2. Preliminaries. Throughout this paper, we assume that p is a prime and $t, d, M, m, k \in N$ where N is the set of natural numbers.

A d-dimensional M-state cellular automaton can be defined as follows. Consider d-dimensional lattice points on which copies of an automaton called "cells" are located. Let $i \in Z^d$ denote the site of a cell, and t discrete time step. Each cell takes a state value which belongs to $\{0, \cdots, M-1\}$. We denote by a_i^t the state of an i-cell at time t. The states of cells at time t are determined by the states of cells at time $t-1$ by,

$$a_i^t = f(a_{i-j^1}^{t-1}, a_{i-j^2}^{t-1}, \cdots, a_{i-j^m}^{t-1})$$

where $j^1, j^2, \cdots, j^m \in Z^d$ and $f:\{0, \cdots, M-1\}^m \to \{0, \cdots, M-1\}$. f is a "local transient function" and $\{j^1, \cdots, j^m\}$ is called a "neighbourhood index".

We treat only the case where f is "linear" i.e.,

$$f(x_1, x_2, \cdots, x_m) = \alpha_1 \cdot x_1 + \cdots + \alpha_m \cdot x_m \bmod M \quad (\alpha_1, \cdots, \alpha_m \in N).$$

The space time pattern of a cellular automaton sometimes reveals interesting properties. To study it in its limit, we define "limit set" of a cellular automaton as follows.

Definition 2.1. Let $T: N \to N$ and $T(1) < T(2) < T(3) \cdots$. We define $S(n)$, $\lim S \subseteq R^{d+1}$ for this $T(n)$ as follows.

$$S(n) = \left\{ \left(\frac{t}{T(n)}, \frac{i_1}{T(n)}, \frac{i_2}{T(n)}, \cdots, \frac{i_d}{T(n)} \right) \middle| a_i^t \neq 0,\ t \leqq T(n) \text{ and } i = (i_1, \cdots, i_d) \right\}.$$

We define limsup $S(n)$ and liminf $S(n)$ as in S. J. Willson ([1], p. 93).

If limsup $S(n)$ = liminf $S(n)$, then we define $\lim S$ the "limit set" of cellular automaton by,

$$\lim S = \limsup S(n) = \liminf S(n).$$

3. Existence of limit set. Unless otherwise stated, we assume that $a_i^0 = 1$ for $i = (0, \cdots, 0)$ and $a_i^0 = 0$ otherwise.

Some properties of linear cellular automata are derivable from those of multinomial coefficients.

Multinomial coefficients have, for example, the following properties