42. *Finite Multiplicity Theorems for Induced Representations of Semisimple Lie Groups and their Applications to Generalized Gelfand-Graev Representations*

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Let G be a connected real semisimple Lie group with finite center, and $\mathfrak{g}$ its Lie algebra. *Generalized Gelfand-Graev representations* (GGGRs), first introduced by Kawanaka [2] for finite reductive groups, form a series of induced representations of G parametrized by nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$. In particular, a principal nilpotent class gives rise to a representation of G induced from a non-degenerate character of a maximal unipotent subgroup. This special type of GGGR, attributed to Gelfand-Graev, is of multiplicity free if G is quasi-split [5].

In this note, we first generalize van den Ban's finite multiplicity theorem [1] for the quasi-regular representation $\mathrm{Ind}_H^G(1_H)$ associated with a semisimple symmetric space G/H, and give nice sufficient conditions for induced representations of G to be of multiplicity finite. Then, applying these criterions, we show that certain interesting types of GGGRs, closely related to the regular representation of G, have finite multiplicity property. Our finite multiplicity theorems are given for *reduced* GGGRs (RGGGRs), a variant of GGGRs. We also give a multiplicity one theorem for RGGGRs under some additional assumptions.

1. Criterions for finite multiplicity property. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and θ the corresponding Cartan involution of $\mathfrak{g}$, which can be lifted up canonically to an involution of G. Denote by K the maximal compact subgroup of G consisting of fixed points of θ on G. Let $Q=LN$ with $L\equiv Q\cap\theta Q$, denote a Levi decomposition of a parabolic subgroup Q of G. Let σ be an involutive automorphism of $\mathfrak{l}\equiv\mathrm{Lie}(L)$ satisfying: (1) σ commutes with $\theta|\mathfrak{l}$, and (2) σ coincides with θ on the split component $\mathfrak{a}$ of $\mathfrak{l}$. Take a closed subgroup H of L with Lie algebra $\mathfrak{h}\equiv\{X\in\mathfrak{l}\,;\,\sigma X=X\}$.

For a continuous representation ζ of the semidirect product subgroup $HN=H\ltimes N$ on a Fréchet space $\mathcal{F}$, we consider the representation C^∞-$\mathrm{Ind}_{HN}^G(\zeta)=(\pi_\zeta, C^\infty(G\,;\zeta))$ of G induced from ζ in C^∞-context: the group G acts on the representation space

$$C^\infty(G\,;\zeta)\equiv\{f\colon G\xrightarrow{C^\infty}\mathcal{F}\,;\,f(gz)=\zeta(z)^{-1}f(g)\ (g\in G,\,z\in HN)\},$$

by left translation. $C^\infty(G\,;\zeta)$ has a $U(\mathfrak{g}_C)$-module structure through differentiation, where $U(\mathfrak{g}_C)$ denotes the enveloping algebra of $\mathfrak{g}_C\equiv\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Let $\mathfrak{Z}$ be the center of $U(\mathfrak{g}_C)$. For an algebra homomorphism $\chi\colon\mathfrak{Z}\to\boldsymbol{C}$, the joint