17. On Weak, Strong and Classical Solutions of the Hopf Equation

An Example of F.D.E. of Second Order

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§1. **Introduction and results.** Let (M, g) be a compact Riemannian manifold of $\dim M=d$ with or without boundary ∂M. We denote by $\mathring{X}_\sigma(M)$ the space of solenoidal vector fields on M which vanish near the boundary. $\boldsymbol{H}$ stands for the completion of the above space with respect to $\boldsymbol{L}^2$-norm, denoted by $|\cdot|$. $\boldsymbol{V}^s$ stands for the completion of $\mathring{X}_\sigma(M)$ in the Sobolev space of order $s \in \boldsymbol{Z}$, whose norm is denoted by $\|\cdot\|_s$. For 1-forms, we introduce $\mathring{\Lambda}^1_\sigma(M)$ analogously. The completions of it with corresponding norms are denoted by $\tilde{\boldsymbol{H}}$ and $\tilde{\boldsymbol{V}}^s$, respectively. The space of symmetric tensor fields with 2 contravariant (or covariant) indices is denoted by $ST_2(M)$ (or $ST^2(M)$.)

Our aim of this paper is to 'solve' the following Functional Derivative Equation (F.D.E.):

(I) Find a functional $W(t, \eta)$, for $t \in (0, \infty)$, $\eta \in \mathring{\Lambda}^1_\sigma(M)$ satisfying

$$\text{(I.1)} \quad \frac{\partial}{\partial t} W(t,\eta)=\int_M \Big[-i\Big\{\frac{\partial}{\partial x^j}\eta_i(x)-\Gamma^l_{ij}(x)\eta_l(x)\Big\}\frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(x)} + \nu(\Delta\eta)_i(x)\frac{\delta W(t,\eta)}{\delta\eta_i(x)}+i\eta_j(x)f^j(x,t)W(t,\eta)\Big]d_g x,$$

$$\text{(I.2)} \quad \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\Big\{\sqrt{g(x)}\frac{\delta W(t,\eta)}{\delta\eta_i(x)}\Big\}=0,$$

$$\text{(I.3)} \quad W(t,0)=1$$

and

$$\text{(I.4)} \quad W(0,\eta)=W_0(\eta).$$

Here $\eta(x)=\eta_j(x)dx^j \in \mathring{\Lambda}^1_\sigma(M)$, and $f(x,t)=f^j(x,t)(\partial/\partial x^j) \in \mathring{X}_\sigma(M)$ for a.e. t, $W_0(\eta)$ is a given positive definite functional on $\mathring{\Lambda}^1_\sigma(M)$ satisfying

$$\text{(I.5)} \quad W_0(0)=1 \quad \text{and} \quad \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^j}\Big\{\sqrt{g(x)}\frac{\delta W_0(\eta)}{\delta\eta_j(x)}\Big\}=0.$$

Hereafter, we use Einstein's convention for contracting indices and also the terminology and symbols from Riemannian geometry and functional analysis. The definition of functional derivatives

$$\frac{\delta W(t,\eta)}{\delta\eta_i(x)} \quad \text{and} \quad \frac{\delta^2 W(t,\eta)}{\delta\eta_i(x)\delta\eta_j(y)}$$

is given, for example, in E. Hopf [3].

A weak solution of Problem (I) will be afforded by considering the