112. *Groups Associated with Compact Type Subalgebras of Kac-Moody Algebras*

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The Kac-Moody groups associated with a given Kac-Moody algebra as constructed by Peterson-Kac [5] have a disadvantage that the exponential map can not be defined on the whole algebra. The present note gives a partial solution to the problem to remedy the situation, by constructing groups in the above title.

§ 1. Kac-Moody algebras. Let $\mathfrak{g}$ be a Kac-Moody algebra and A the corresponding generalized Cartan matrix (GCM). Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, Δ the root system of $(\mathfrak{g}, \mathfrak{h})$, Π the set of simple roots, Δ_+ the set of positive roots with respect to Π, and W the Weyl group. We denote by $\mathfrak{g}_{\boldsymbol{R}}$ the Kac-Moody algebra over the real number field $\boldsymbol{R}$ corresponding to the GCM A, and by $\mathfrak{h}_{\boldsymbol{R}}$ the Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{R}}$. Then, $\mathfrak{g}=\boldsymbol{C}\otimes\mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{h}=\boldsymbol{C}\otimes\mathfrak{h}_{\boldsymbol{R}}$. There exists an involutive antilinear automorphism ω_0 on $\mathfrak{g}$ such that

(1.1) $$\omega_0(h)=-h \quad (h\in\mathfrak{h}_{\boldsymbol{R}}), \qquad \omega_0(\mathfrak{g}^\alpha)=\mathfrak{g}^{-\alpha} \quad (\alpha\in\Delta),$$

where $\mathfrak{g}^\alpha$ is the α-root space (cf. [3, Chap. 2]). We denote by $\mathfrak{k}$ and $\mathfrak{k}_{\boldsymbol{R}}$ the set of fixed points of ω_0 in $\mathfrak{g}$ and $\mathfrak{g}_{\boldsymbol{R}}$ respectively. Then, $\mathfrak{k}_{\boldsymbol{R}}=\mathfrak{k}\cap\mathfrak{g}_{\boldsymbol{R}}$. Since ω_0 is an involution, $\mathfrak{k}$ is a real form of $\mathfrak{g}$ as a Lie algebra. We call $\mathfrak{k}$ the *unitary form* of $\mathfrak{g}$ and $\mathfrak{k}_{\boldsymbol{R}}$ a *compact type subalgebra* of $\mathfrak{g}_{\boldsymbol{R}}$. If $\mathfrak{g}$ is finite-dimensional, then $\mathfrak{g}$ is semisimple, $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$, and $\mathfrak{k}_{\boldsymbol{R}}$ is a maximal compact subalgebra of $\mathfrak{g}_{\boldsymbol{R}}$.

We assume throughout that the GCM A is symmetrizable (cf. [3]). Then, there exists a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$, a standard invariant form, which is infinitesimally invariant under ad $\mathfrak{g}$. The restriction of $(\cdot|\cdot)$ to $\mathfrak{h}$ is W-invariant and non-degenerate, and defines a W-equivariant linear bijection ν from $\mathfrak{h}$ onto its dual $\mathfrak{h}^*$. We denote by the same symbol $(\cdot|\cdot)$ the induced bilinear form on $\mathfrak{h}^*$. Then we have

(1.2) $$[x, y]=(x|y)\nu^{-1}(\alpha) \qquad (x\in\mathfrak{g}^\alpha,\ y\in\mathfrak{g}^{-\alpha},\ \alpha\in\Delta).$$

We define a sesquilinear form $(\cdot|\cdot)_0$ on $\mathfrak{g}$ as

(1.3) $$(x|y)_0=-(x|\omega_0(y)) \qquad (x, y\in\mathfrak{g}).$$

Then, according to [4, Theorem 1], $(\cdot|\cdot)_0$ is Hermitian and its restriction to each root space $\mathfrak{g}^\alpha$ is positive definite.

Put $\mathfrak{n}_\pm=\sum_{\alpha\in\Delta_+}\mathfrak{g}^{\pm\alpha}$. Then, they are both subalgebras of $\mathfrak{g}$, and we have a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ (direct sum).

§ 2. Irreducible highest weight modules. Let $\lambda\in\mathfrak{h}^*$ and L_λ be the left ideal of the enveloping algebra $U(\mathfrak{g})$ generated by $\mathfrak{n}_+$ and $\{h-\lambda(h)\,|\,h\in\mathfrak{h}\}$.