37. *Density of the Range of a Wave Operator with Nonmonotone Superlinear Nonlinearity*

By Kazunaga TANAKA
Department of Mathematics, Nagoya University

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1. Introduction. In this article we shall study the nonlinear wave equation:

(1) $u_{tt}-u_{xx}+g(u)=f(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R},$

(2) $u(0,t)=u(\pi,t)=0, \qquad t\in\boldsymbol{R},$

(3) $u(x,t+T)=u(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R},$

where $T>0$ is a rational multiple of π, $g(s)$ is a continuous function on $\boldsymbol{R}$ and $f(x,t)$ is a given T-periodic function of t.

Many mathematicians concerned with this problem (see [1], [7] and its references). Except for [2, 3, 6, 11, 12] they ask that $g(s)$ is monotonic, in order to overcome the lack of compactness due to the fact that the kernel of the wave operator $\partial_t^2-\partial_x^2$ is infinite dimensional.

Working in a restricted class $\tilde{H}$ of functions satisfying some symmetry properties and such that

(i) $\tilde{H}\cap\operatorname{Ker}(\partial_t^2-\partial_x^2)=\{0\}$,

(ii) $\tilde{H}$ is invariant under $\partial_t^2-\partial_x^2$ and g,

J. M. Coron [3] proved the existence of multiple T-periodic solutions of (1)–(3) in case $f\equiv 0$ and the existence of forced vibrations under the condition $f\in\tilde{H}$ without assumption of monotonicity. See also N. Basile and M. Mininni [2].

On the other hand, M. Willem [11, 12] and H. Hofer [6] also dealt with the problem (1)–(3) without the monotonicity assumption. They tackled the infinite dimensional kernel of $\partial_t^2-\partial_x^2$ without introducing restricted classes. Under the following *nonresonance* condition:

For consecutive eigenvalues $\alpha<\beta$ of $-(\partial_t^2-\partial_x^2)$ and

(4) for some constants $\varepsilon>0$, $r>0$,

$$\alpha+\varepsilon\leqq\frac{g(s)}{s}\leqq\beta-\varepsilon \qquad \text{for } |s|\geqq r,$$

and some additional conditions, they proved that (1)–(3) is *almost solvable*; (1)–(3) possesses a solution for a *dense set* of f's in L^2, in other words, the range of the operator: $u\to u_{tt}-u_{xx}+g(u)$ is dense in L^2. Their arguments are based on the variational methods; [11, 12] used I. Ekeland's variational principles (c.f. [4]), [6] used Leray-Schauder theory in conjunction with the variational method. Note that under the condition (4) the solutions of (1)–(3) are a priori bounded in L^2. See also K. Tanaka [10].

This paper is an extension of [6, 10, 11, 12] and deals with the case