90. The Number of Embeddings of Integral Quadratic Forms. I*)

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1. Introduction. An *integral quadratic form* is a free Z-module L of finite rank together with a map $q: L \to Z$ such that the induced map $b: L \times L \to Z$ defined by $b(x, y) = q(x+y) - q(x) - q(y)$ is Z-bilinear, and such that $q(nx) = n^2 q(x)$ for all $n \in Z$ and $x \in L$. (This is sometimes called an "even" form in the literature, since the function $x \mapsto b(x, x) = 2q(x)$ assumes only even values.) The adjoint map of the associated bilinear form b is a Z-linear map $\mathrm{Ad}\, b: L \to L^{\#} = \mathrm{Hom}\,(L, Z)$; L is called *nondegenerate* if $\mathrm{Ad}\, b$ is injective, and *unimodular* if $\mathrm{Ad}\, b$ is an isomorphism.

If M and L are integral quadratic forms, an *embedding* of M into L is an injective homomorphism of Z-modules $\phi: M \to L$ which preserves the quadratic maps; ϕ is called *primitive* if $\mathrm{coker}\, \phi$ is free. Nikulin [3] has given necessary and sufficient conditions for the existence of a primitive embedding of M into L in the case that M is nondegenerate and L is indefinite and unimodular.

A Z-module isomorphism $\sigma: M \to L$ which preserves the quadratic maps is called an *isometry*. The group of all isometries from L to itself is denoted by $O(L)$. We say that two primitive embeddings $\phi_1, \phi_2: M \to L$ are *equivalent* if there is an isometry σ in $O(L)$ such that $\sigma \circ \phi_1 = \phi_2$. (There are also some restricted notions of equivalence in which σ is required to lie in a specified subgroup of $O(L)$.) Our goal is to count the number of equivalence classes of primitive embeddings from a nondegenerate M into an indefinite unimodular L, assuming that one such embedding exists. In this note, we modify some arguments of Nikulin [3] and Wall [4] to express this number in terms of a certain invariant of the orthogonal complement N of M in L; a subsequent note will give a procedure for computing that invariant when N is indefinite and has rank at least three. The proofs, together with some applications to algebraic geometry, will be given elsewhere.

2. Real quadratic forms and subgroups of the orthogonal group. Let L be a nondegenerate integral quadratic form. If we extend q to a map $q: L \otimes R \to R$ by the requirement $q(rx) = r^2 q(x)$ for $r \in R$ and $x \in L$, then

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