74. *On an Euler Product Ring*

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§1. **Euler product rings.** Let Z be the ring of rational integers. We denote by $E(Z)$ the (universal) completion $\hat{Z}$ of Z. Hence, denoting the ring of p-adic integers by Z_p, we have a canonical isomorphism $E(Z) \cong \prod_p Z_p$, where p runs over all rational primes. We consider $E(Z)$ as an "Euler product ring" (over Z) via this infinite product expression; see Theorem 1 below for another explanation. In this paper we note some properties of $E(Z)$ related to the structure of maximal ideals of $E(Z)$ in a bit generalized situation. A detailed study will appear elsewhere.

We fix the notation. Let A be a commutative ring with 1. We define: $E(A) = A \otimes_Z E(Z)$. We denote by $\mathrm{Max}(A)$ the space of all maximal ideals of A, which is equipped with the Stone topology. For $q \in \mathrm{Max}(Z) \cup \{0\}$ we put

$$\mathrm{Max}_q(A) = \{M \in \mathrm{Max}(A);\ \text{the characteristic of } A/M \text{ is } q\}.$$

We say that $M \in \mathrm{Max}(A)$ is *cofinite* if A/M is a finite field, and define the norm $N(M)$ of M via $N(M) = \sharp(A/M)$, where $\sharp$ denotes the cardinality. We denote by $\mathrm{Max}^{cf}(A)$ the set consisting of all cofinite maximal ideals of A. Obviously we have:

$$\mathrm{Max}^{cf}(A) \subset \mathrm{Max}(A) - \mathrm{Max}_0(A) = \mathrm{Max}_2(A) \cup \mathrm{Max}_3(A) \cup \cdots.$$

We define the zeta function $\zeta(s, A)$ of A (at least formally) by the following Euler product $\zeta(s, A) = \prod_M (1 - N(M)^{-s})^{-1}$ where M runs over $\mathrm{Max}^{cf}(A)$ and s is a complex number; this zeta function coincides with the zeta function $\zeta(s, M(A))$ of the category $M(A)$ of A-modules in the sense of [5]. (We note that some details of [5] are appearing in Proc. London Math. Soc.) We denote by $\Omega(A)$ the A-module of absolute Kähler differentials of A (over Z); we refer to Grothendieck [2; Chap. 0, §20] concerning Kähler differentials.

Hereafter, let $A = O_F$ be the integer ring of a finite number field F. Then $E(A) \cong \hat{A} \cong \prod_{\mathfrak{p}} A_{\mathfrak{p}}$, where $\hat{A}$ and $A_{\mathfrak{p}}$ denote respectively the completion and $\mathfrak{p}$-adic completion of A, and $\mathfrak{p}$ runs over $\mathrm{Max}(A)$. We have:

Theorem 1. $\zeta(s, E(A)) = \zeta(s, A)$.

Theorem 2. *$\mathrm{Max}(E(A))$ is a compact Hausdorff space.*

Theorem 3. $\Omega(E(A)) \neq 0$.

Remark 1. (1) $\zeta(s, A)$ is equal to the Dedekind zeta function of F. (2) $\mathrm{Max}(A)$ is not a Hausdorff space. (3) $\Omega(A) = 0$.

§2. **Proofs.** First we show

Theorem 1a. $\mathrm{Max}_p(E(A)) = \{\mathfrak{p}E(A);\ \mathfrak{p} \in \mathrm{Max}(A), \mathfrak{p}|p\}$ *for each rational*