62. A Note on the Mean Value of the Zeta and L-functions. I

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1. The aim of the present series of notes is to develop a study on the various mean values of the Riemann zeta- and Dirichlet L-functions; here, to begin with, we investigate the square mean of L-functions viewing it as a generalization of the situation considered by Atkinson [1].

Let χ be a Dirichlet character, and put, for two complex variables u and v

$$Q(u, v; q)=\frac{1}{\varphi(q)}\sum_{\chi \pmod q} L(u, \chi)L(v, \bar{\chi}),$$

where $q\geqq 2$ and φ is the Euler function. If $Re(u)>1$, $Re(v)>1$, then

$$Q(u, v, q)=L(u+v, \chi_0)+f(u, v; q)+f(v, u; q), \tag{1}$$

where χ_0 is the principal character mod q, and

$$f(u, v; q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(qm+a)^{-u}(q(m+n)+a)^{-v}.$$

We need an analytic continuation of $f(u, v; q)$ valid when $Re(u)<1$, $Re(v)<1$. This may be obtained by Poisson's summation formula as in [1], but we take an alternative way which starts from the following integral representation: When $Re(u)>0$, $Re(v)>1$, $Re(u+v)>2$,

$$f(u, v; q)=\frac{q^{-u-v}}{\Gamma(u)\Gamma(v)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_0^\infty \frac{y^{v-1}}{e^y-1}\int_0^\infty \frac{e^{(a/q)(x+y)}}{e^{x+y}-1}x^{u-1}dxdy.$$

To remove the singularity at $x+y=0$ we put

$$h(z; q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{e^{(a/q)z}}{e^z-1}-\frac{1}{z}\right),$$

and note that when $0<Re(u)<1$ and $y>0$

$$\int_0^\infty x^{u-1}(x+y)^{-1}dx=y^{u-1}\Gamma(u)\Gamma(1-u).$$

Then, we find that when $0<Re(u)<1$, $Re(u+v)>2$,

$$\begin{aligned}(2)\quad & f(u, v; q)\\ &=\varphi(q)q^{-(u+v)}\Gamma(u+v-1)\Gamma(1-u)\{\Gamma(v)\}^{-1}\zeta(u+v-1)+g(u, v; q),\end{aligned}$$

where

$$g(u, v; q)=\frac{q^{-u-v}}{\Gamma(u)\Gamma(v)}\int_0^\infty \frac{y^{v-1}}{e^y-1}\int_0^\infty h(x+y; q)x^{u-1}dxdy.$$

Next we introduce the contour C which starts at infinity, proceeds along the positive real axis to δ $(0<\delta<1/2)$, describes a circle of radius δ counterclockwise round the origin and returns to infinity along the positive real axis; we have, for $0<Re(u)<1$, $Re(u+v)>2$,