55. A Note on Jacobi's Generating Function for the Jacobi Polynomials

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Some elementary identities in the theory of the Gaussian hypergeometric series are used here to present a simple proof of Jacobi's generating function for the Jacobi polynomials.

In the literature there are several interesting proofs of Jacobi's generating function for the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta}, \tag{1}$$

where $R=(1-2xt+t^2)^{1/2}$. See, for example, Szegö [6, Section 4.4], Rainville [4, Section 140], Carlitz [2], Askey [1], and Foata and Leroux [3]; see also Srivastava and Manocha [5, p. 82]. We give here a simple proof which uses the definition [6, p. 62, Equation (4.21.2)]

$$P_n^{(\alpha,\beta)}(x) = \begin{bmatrix}\alpha+n\\ n\end{bmatrix} {}_2F_1\begin{bmatrix}-n,\ \alpha+\beta+n+1; & \\ & \dfrac{1-x}{2}\\ \alpha+1; & \end{bmatrix} = \sum_{k=0}^{n}\begin{bmatrix}\alpha+n\\ n-k\end{bmatrix}\begin{bmatrix}\alpha+\beta+n+k\\ k\end{bmatrix}\left[\frac{x-1}{2}\right]^k, \tag{2}$$

and such elementary results from the theory of the Gaussian hypergeometric series ${}_2F_1$ as the transformation [4, p. 60, Equation (4)]

$${}_2F_1\begin{bmatrix}a,\ b; & \\ & z\\ c; & \end{bmatrix} = (1-z)^{-a}\,{}_2F_1\begin{bmatrix}a,\ c-b; & \\ & -\dfrac{z}{1-z}\\ c; & \end{bmatrix}, \tag{3}$$

the reduction formula [4, p. 70, Problem 10]

$${}_2F_1\begin{bmatrix}a,\ a+\frac{1}{2}; & \\ & z\\ 2a; & \end{bmatrix} = \frac{1}{\sqrt{1-z}}\left[\frac{1+\sqrt{1-z}}{2}\right]^{1-2a}, \tag{4}$$

and the binomial expansion [4, p. 58, Equation (1)]

$${}_2F_1\begin{bmatrix}a,\ b; & \\ & z\\ b; & \end{bmatrix} = \sum_{n=0}^{\infty}\begin{bmatrix}a+n-1\\ n\end{bmatrix}z^n = (1-z)^{-a}. \tag{5}$$

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