

45. The Poincaré Lemma for a Variation of Polarized Hodge Structure

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0. The purpose of this note is to present the Poincaré lemma for a variation of Hodge structure. The result was first proved in a pioneering work of Zucker ([4]) in the one-dimensional case. Cattani and Kaplan [1] has recently announced a generalization in the case of dimension 2 and weight 1.

Our main result (Theorem 1) shows the coincidence of the intersection cohomology groups and the L^2 -cohomology groups associated with a variation of Hodge structure in higher dimensional case, thus generalizing the Poincaré lemma due to Zucker to the higher dimensional case. Note, however, that the holomorphic Poincaré lemma in the sense of [4] does not hold in the higher dimensional case. (See [1].) The proof of Theorem 1 is based on an algebraic result (Theorem 2), which is announced by Cattani and Kaplan [1] in the two-dimensional case.

1. Let X be an n -dimensional complex manifold, Y a normally crossing hypersurface and (H_Z, F, S) a variation of polarized Hodge structure of weight w over $X \setminus Y$, that is, H_Z is a local system on $X \setminus Y$, S is a non-degenerate bilinear form on H_Q and F is a finite filtration of $\mathcal{O}_{X \setminus Y} \otimes H_Z$ by holomorphic vector bundles such that at any point x in $X \setminus Y$ the stalk of (H_Z, F, S) gives a polarized Hodge structure and $vF^p \subset F^{p-1}$ for any holomorphic vector field v and any p . Then H gives a C^∞ -vector bundle on $X \setminus Y$ with the Hermitian metric given by the polarization.

2. Let us take a Riemannian metric g on $X \setminus Y$ which behaves on a neighborhood of Y as follows:

For any point y_0 of Y let us take a local coordinate system (z_1, \dots, z_n) such that Y is defined by $z_1 \cdots z_l = 0$. Then we assume

$$g \sim \sum_{j \leq l} \frac{dz_j d\bar{z}_j}{(|z_j| \log |z_j|)^2} + \sum_{j > l} dz_j d\bar{z}_j.$$

Here \sim means that each of the two metrics is bounded by a constant multiple of the other on a neighborhood of y_0 . One can easily show (see [4]) that such a metric exists. If X is a Kähler manifold, we can choose a Kähler metric as g .

3. Let us define the sheaf $\mathcal{D}^p(H)$ on X as follows:

For any open set U of X , $\Gamma(U, \mathcal{D}^p(H))$ is the set of distribution-valued p -forms with coefficients in H defined on $U \setminus Y$.

We also define the subsheaf $\mathcal{L}^p(H)_{(2)}$ of $\mathcal{D}^p(H)$ as follows: