44. The Riemann-Roch Theorem and Bernoulli Polynomials

By Tetsuya ANDO
Department of Mathematics, Faculty of Science,
University of Tokyo

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0. Introduction. Let X be a non-singular algebraic variety with $\dim X=N$ over an algebraically closed field. In this paper we shall prove the following formula

$$\chi(tK_X)=\sum_{r=0}^{[N/2]}\frac{\phi_{N-2r}(t)}{(N-2r)!}K_X^{N-2r}R_r.$$

Here the $\phi_n(t)$ denote the Bernoulli polynomials, defined by

$$\frac{xe^{tx}}{e^x-1}=\sum_n\frac{\phi_n(t)}{n!}x^n,$$

$R_n=R_n(c_1,\cdots,c_{2n})$ is a polynomial of Chern classes, defined by

$$T_{2n+1}(c_1,\cdots,c_{2n})=(1/2)c_1R_n(c_1,\cdots,c_{2n})$$

where T_r is the r-th todd class of X.

1. Preliminaries. We start by recalling the following elementary facts.

Lemma 1.

(1-1) $$\phi_0(t)=1,\qquad \phi_1(t)=t-(1/2).$$

(1-2) $$(d/dt)\phi_n(t)=n\cdot\phi_{n-1}(t).$$

(1-3) $$\phi_{2n+1}(0)=\phi_{2n+1}(1/2)=0\qquad \textit{for } n\geqq 1.$$

(1-4) $$\phi_n(t+1)-\phi_n(t)=nt^{n-1}.$$

(1-5) $$\phi_n(t)=\sum_{r=0}^{n}\binom{n}{r}\phi_r(0)t^{n-r},\qquad \phi_{2n}(t)=\sum_{r=0}^{m}\binom{2m}{2r}\phi_{2r}(0)t^{2m-2r}-mt^{2m-1}.$$

(1-6) $$\sum_{r=0}^{m}\binom{2m}{2r}\frac{2^{2r}\phi_{2r}(0)}{2m-2r+1}=1.$$

Proof. We only prove (1-6). From (1-5) we have

$$\frac{\phi_{2m+1}(t)}{2m+1}=\sum_{r=0}^{m}\binom{2m}{2r}\frac{\phi_{2r}(0)}{2m-2r+1}t^{2m-2r+1}-\frac{1}{2}t^{2m}.$$

Put $t=1/2$. Then

$$0=\sum_{r=0}^{m}\binom{2m}{2r}\frac{\phi_{2r}(0)}{2m-2r+1}\cdot\frac{1}{2^{2m-2r+1}}-\frac{1}{2^{2m+1}}.$$

From this (1-6) follows. Q.E.D.

We define the symbols $c_1,\cdots,c_N$; $p_1,\cdots,p_N$; $z_1,\cdots,z_N$; $x_1,\cdots,x_N$; and polynomials $A_i(p_1,\cdots,p_i)$, $T_i(c_1,\cdots,c_i)$ $(0\leqq i\leqq N)$ and $R_j(c_1,\cdots,c_{2j})$ $(0\leqq j\leqq[N/2])$ as follows:

(1) $$z_i=x_i^2\qquad \text{for } 1\leqq i\leqq N.$$

(2) p_i is the i-th elementary symmetric function of $x_1,\cdots,x_N$.

(3) c_i is the i-th elementary symmetric function of $z_1,\cdots,z_N$.