39. *Polynomial Difference Equations which have Entire Solutions of Finite Order*

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1. Introduction. Here we consider the difference equation

$$y(x+1)^m = a_p y(x)^p + a_{p-1} y(x)^{p-1} + \cdots + a_1 y(x) + a_0, \tag{1.1}$$

where $a_p, a_{p-1}, \cdots, a_1, a_0$ are constants, $a_p \neq 0$.

When $m=1$, the equation (1.1) has been studied by several authors [1], [4], [5]. We consider here mainly the case $m \geqq 2$.

We proved in [2] the following theorem.

Theorem A. *Let $R_j(x, w)$, $j=0, 1$, be rational functions:*

$$R_j(x, w) = P_j(x, w)/Q_j(x, w),$$
$$P_j(x, w) = a_{p_j}^{(j)}(x) w^{p_j} + \cdots + a_0^{(j)}(x),$$
$$Q_j(x, w) = b_{q_j}^{(j)}(x) w^{q_j} + \cdots + b_0^{(j)}(x),$$

in which $a_k^{(j)}(x)$ and $b_h^{(j)}(x)$, $k=0, \cdots, p_j$, $h=0, \cdots, q_j$, $j=0, 1$, are polynomials, $a_{p_j}^{(j)}(x) b_{q_j}^{(j)}(x) \not\equiv 0$. Consider the difference equation

$$R_1(x, y(X+1)) = R_0(x, y(x)). \tag{1.2}$$

Suppose (1.2) *possesses a meromorphic solution $y(x)$, which is of finite order. Then, either $y(x)$ is rational, or there holds*

$$\max(p_1, q_1) = \max(p_0, q_0).$$

By this theorem, we know that the equation (1.1) admits a meromorphic solution of finite order only if

$$m = p.$$

In particular, when $m=1$, it is easy to see that (1.1) admits an entire solution of finite order if $p=1$. Our aim in this note is to determine the form of the equations (1.1) which have entire solutions of finite order, when $m \geqq 2$. Our results are as follows.

Theorem 1. *The equation* (1.1) *possesses an entire nontrivial solution of finite order if and only if it is either of the form*

$$m \text{ is even and } y(x+1)^m = (A^2 - y(x)^2)^{m/2},\ A \neq 0, \tag{1.3}$$

i.e.,

$$y(x+1)^2 = A^2 - y(x)^2, \tag{1.3'}$$

or of the form

$$y(x+1)^m = (a y(x) + b)^m. \tag{1.4}$$

By the way, we note that the equation (1.3) is satisfied by

$$y(x) = A \sin(\pi x/2) \quad \text{and} \quad y(x) = A \cos(\pi x/2).$$

The proof of Theorem 1 is implied in the following lemmas.

Lemma 2. *The equation* (1.1) *can not have an entire nontrivial solu-*