70. General Solutions of Witten's Gauge-field Equations

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§0. Introduction. Consider a gauge field in the eight-dimensional space satisfying

$$(1)\quad \begin{aligned} &[-\lambda_1\nabla_{\eta_1}+\nabla_{\zeta_1},\ \lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1}]=0, \qquad [-\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2},\ \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ -\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}]=[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[\lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ -\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}]=[\lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\end{aligned}$$

for any $\lambda_1, \lambda_2 \in \boldsymbol{C}$. Here $(\eta_1, \bar\eta_1, \zeta_1, \bar\zeta_1, \eta_2, \bar\eta_2, \zeta_2, \bar\zeta_2) \in \boldsymbol{C}^8$ and ∇_{η_1} etc. are covariant derivatives. To the gauge fields satisfying (1) corresponds a class of solutions of the Yang-Mills equations including all the self-dual or anti-self-dual solutions, when they are restricted to the four-dimensional subspace $\eta_1-\eta_2=\bar\eta_1-\bar\eta_2=\zeta_1-\zeta_2=\bar\zeta_1-\bar\zeta_2=0$ (cf. [1], [2]).

In our previous paper [2], we regarded (1) as the integrability condition for some linear equations with a pair of spectral parameters λ_1, λ_2. But in this paper, we note that the equations (1) imply

$$(2)\quad \begin{aligned} &[-\lambda\nabla_{\eta_1}+\nabla_{\zeta_1},\ \lambda\nabla_{\zeta_1}+\nabla_{\bar\eta_1}]=0, \qquad [-\lambda\nabla_{\eta_2}+\nabla_{\zeta_2},\ \lambda\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[-\lambda\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ -\lambda\nabla_{\eta_2}+\nabla_{\zeta_2}]=[-\lambda\nabla_{\eta_1}+\nabla_{\bar\zeta_1},\ \lambda\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[\lambda\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ -\lambda\nabla_{\eta_2}+\nabla_{\zeta_2}]=[\lambda\nabla_{\zeta_1}+\nabla_{\bar\eta_1},\ \lambda\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\end{aligned}$$

for any $\lambda \in \boldsymbol{C}$. (This system of equations is classified into the class A_4 according to R. S. Ward [3].)

Therefore, the solution space of (1) is embedded in the solution space of (2), and it is sufficient to solve the following problems:

(i) to construct the general solutions of (2) and clarify their structure,

(ii) to characterize the solutions of (1) in the solution space of (2).

The problem (i) can be solved more simply than to investigate directly the solution space of (1) which we did in our previous paper [2], because the equations (2) comprise only one spectral parameter. In fact, we can solve them by direct application of Sato-Takasaki method (cf. [4], [5], [6]): we rewrite (2) into a system of differential equations for unknown functions valued in an infinite-dimensional Grassmann manifold and investigate the structure of its solution space by considering an initial-value problem with respect to the subspace $\bar\zeta_1=\bar\eta_1=\zeta_2=\bar\eta_2=0$.

The problem (ii) also can be solved and a simple characterization can be obtained. (See Theorem 3.)