127. On Powers of the Denominators of Rationals

By Chiaki NAGASAKA
Department of Mathematics, Kyushu University

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For a rational x let $\operatorname{den} x$ mean its reduced denominator >0. We denote by $\operatorname{den}^{\alpha} x$ the αth power of $\operatorname{den} x$, where α is a complex number. In this note we show some properties of the function $\operatorname{den}^{\alpha} x$.

Theorem 1. *Let a, b, c, d be rational integers with $ad-bc=1$. Then for any rational x with $cx+d\neq 0$, we have*

$$\operatorname{den}^{\alpha} \frac{ax+b}{cx+d}=|cx+d|^{\alpha} \operatorname{den}^{\alpha} x. \tag{1}$$

Conversely, if a function $f(x)$ defined on the rationals satisfies the functional equations

$$f(x+1)=f(x) \tag{*}$$

$$f\left(-\frac{1}{x}\right)=|x|^{\alpha} f(x) \qquad (x\neq 0), \tag{**}$$

then

$$f(x)=f(0) \operatorname{den}^{\alpha} x.$$

Theorem 2. *For every positive integer n, we have*

$$n^{\alpha-1} \sum_{\substack{ad=n \\ d>0}} \sum_{b=0}^{d-1} d^{-\alpha} \operatorname{den}^{\alpha} \frac{ax+b}{d}=\frac{1}{n}\sigma_{\alpha+1}(n) \operatorname{den}^{\alpha} x, \tag{2}$$

where $\sigma_{\alpha+1}(n)$ is the sum of the $(\alpha+1)$th powers of positive divisors of n.

Theorem 3. *For every positive integer n, we have*

$$\sum_{b=0}^{n-1} \operatorname{den}^{\alpha} \frac{x+b}{n}=\sum_{d|n} \mu(d)d^{\alpha}\sigma_{\alpha+1}\left(\frac{n}{d}\right) \operatorname{den}^{\alpha} dx, \tag{3}$$

where μ means Möbius' function.

Proof of Theorem 1. The transformation property (1) is easily verified. We prove the second assertion by induction on $\operatorname{den} x$. Write x in the form h/k where h and k are relatively prime integers and $k>0$. When $\operatorname{den} x=k=1$, namely x is a rational integer, we see immediately by the equation $(*)$

$$f(x)=f(0)=f(0) \operatorname{den}^{\alpha} x.$$

When $\operatorname{den} x=k>1$, we assume that our assertion is valid for any rational with the reduced denominator less than k. We may consider $0<x<1$ because of $(*)$. Then, using the equation $(**)$, we find

$$f(x)=f\left(\frac{h}{k}\right)=f\left(-\frac{1}{-k/h}\right)=\left|-\frac{k}{h}\right|^{\alpha} f\left(-\frac{k}{h}\right)$$
$$=\left(\frac{k}{h}\right)^{\alpha} h^{\alpha} f(0)=k^{\alpha} f(0)=f(0) \operatorname{den}^{\alpha} x$$