78. On the Borel Summability of $\sum_{n=1}^{\infty} n^{-\alpha} \exp(in^{\beta}\theta)$

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0. In the previous paper [1] we proved that $\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^{\beta}\theta)$ is convergent if $1<\beta<2\alpha$. We now consider the Borel summability and apply a Tauberian theorem (cf. [2] Theorem 156) to show that this series converges for all $\theta>0$ when $\alpha=1/2$, $\beta<5/4$.

In fact we prove

Theorem (cf. [2] Notes, Chapter IX). *If $\beta<\alpha+3/4$ then*

$$\sum_{n=1}^{\infty} n^{-\alpha}\exp(in^{\beta}\theta) \qquad (i^2=-1) \tag{1}$$

is Borel summable for all $\theta>0$.

Corollary. *If $\alpha\geqq 1/2$ and $1<\beta<5/4$, then* (1) *converges for all $\theta>0$.*

1. Main lemma. For $1<\beta<3/2$, $\theta>0$ and positive integers m, set

$$F(t)=\left(\frac{\beta}{2}\right)\theta m^{\beta-2}t^2+\beta\theta m^{\beta-1}t=At^2+Bt, \quad \text{say.} \tag{2}$$

Let μ be such that as $m\to\infty$,

$$\mu/m^{1/2+\delta}\longrightarrow 1 \qquad \left(0<\delta<\frac{1}{2}-\frac{\beta}{3}\right). \tag{3}$$

Lemma.

$$\int_{-\mu}^{\mu}\frac{t}{m}\exp\left(-\frac{t^2}{2m}\right)\sin(F(t)-2k\pi t)dt=O\left(\frac{m^{-1/4+(3/2)\delta}}{\sqrt{|k-B/2\pi|}}\right), \tag{4}$$

where k is a positive integer, and $B/2\pi \notin Z$.

Proof. We write

$$(4)=m^{-1}\left(\int_{-\mu}^{0}+\int_{0}^{\mu}\right)=m^{-1}(J_1+J_2), \quad \text{say.}$$

We only consider J_2 here since J_1 similarly estimated.

By changing variable and by the second mean value theorem, we have

$$J_2=\frac{1}{2m}\int_0^{\xi}\sin(Au+(B-2k\pi)\sqrt{u})du.$$

Then by van der Corput's lemma (cf. [3] Lemma 4.4), we have

$$J_3=O\left(m^{-1/4+(3/2)\delta}\Big/\sqrt{\left|k-\frac{B}{2\pi}\right|}\right), \qquad \text{where } \frac{B}{2\pi}\notin Z.$$