53. A Generalization of the Fenchel-Moreau Theorem

By Shozo KOSHI and Naoto KOMURO
Department of Mathematics, Hokkaido University

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1983)

1. Let F be a real valued convex function defined on a locally convex space. The Fenchel-Moreau theorem is that $F(x)=F^{**}(x)$ if and only if F is lower semi-continuous at x ([1]). Many authors considered to generalize this theorem when F is a convex operator defined on a topological linear spaces to Riesz spaces. For example, J. Zowe has proved $F(x)=F^{**}(x)$ if F is continuous at x and x is an interior point of the domain of F. We shall consider the theorem in the case where F is not necessary continuous, nor the interior of the domain is non-empty. In the following, let X and Y be two Hausdorff locally convex topological vector spaces and Y is assumed further a Dedekind complete Riesz space (order complete vector lattice).

To relate the order structure and the topological structure, we demand furthermore that the linear topology of Y is *normal* i.e. the family of the following sets

$$(V+Y^{+})\cap(V-Y^{+})\,;\ V \text{ is an open set containing } 0,$$

constitutes a base of neighbourhoods of 0 for Y, where Y^{+} denotes the totality of elements of Y equal to or greater than 0. A convex operator F defined on X into Y is to mean that the domain of F (denoted by $D(F)$) is a non-empty convex subset of X and

$$F(\alpha x_1+\beta x_2)\leqq \alpha F(x_1)+\beta F(x_2)$$

for $\alpha+\beta=1$ ($\alpha,\beta\in[0,1]$) and $x_1, x_2\in D(F)$.

We shall define the conjugate function F^{*} of F. Let $L(X, Y)$ be the totality of all continuous linear operator from X to Y. For $A\in L(X, Y)$, we define F^{*} as follows:

$$F^{*}(A)=\sup\{A(x)-F(x)\,;\ x\in D(F)\}.$$

It is easy to see that F^{*} is a convex operator from $L(X, Y)$ to Y. Similarly, considering $X\subset L(L(X, Y), Y)$, we can define the double conjugate of F:

$$F^{**}(x)=\sup\{A(x)-F^{*}(A)\,;\ A\in L(X, Y)\}.$$

As usual, we define the subdifferential of F at $x\in D(F)$ with

$$\partial F(x)=\{A\in L(X, Y)\,;\ A(x)-A(x')\geq F(x)-F(x'),\ x'\in D(F)\}.$$

Furthermore, we shall define the y-subdifferential for $y\in Y^{+}$ as follows:

$$\partial_y F(x)=\{A\in L(X, Y)\,;\ A(x)-A(x')\geq F(x)-F(x')-y,\ x'\in D(F)\}.$$

It is easy to see that

(a) $\partial F(x)\neq\phi$ implies $\partial_y F(x)\neq\phi$,