104. Certain Irreducible Polynomials with Multiplicatively Independent Roots

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(Communicated by Kunihiko KODAIRA, M. J. A., Nov. 12, 1982)

§1. **Statement of the results.** For an integer $k\geq 3$, let us define a polynomial $P_{k,p}$ of degree $p\geq 4$:

$$P_{k,p}(x)=x^p+k(x^{p-1}+x^{p-2}+\cdots+x)+1.$$

In this note, we prove the following theorem.

Theorem. 1. *For even p, $P_{k,p}(x)$ is irreducible over $\boldsymbol{Z}$. For odd p, $(x+1)^{-1}P_{k,p}(x)$ is irreducible over $\boldsymbol{Z}$.*

2. *The polynomial has the following decompositions.*

$$P_{k,p}(x)=(x+\alpha)(x+\alpha^{-1})\prod_{i=1}^{p/2-1}(x-\varepsilon_i)(x-\bar{\varepsilon}_i) \qquad \textit{for even } p$$

$$=(x+1)(x+\alpha)(x+\alpha^{-1})\prod_{i=1}^{(p-1)/2-1}(x-\varepsilon_i)(x-\bar{\varepsilon}_i) \qquad \textit{for odd } p$$

where α is a real number such that $0<|\alpha-k+1|<(k-1)^{-(p-3)}$ and $|\varepsilon_i|=1$, $i=1,\cdots,[p/2]-1$. Here $\bar{\varepsilon}$ means the complex conjugate of ε and $|\varepsilon|=\sqrt{\varepsilon\bar{\varepsilon}}$.

3. *The roots $\alpha,\varepsilon_1,\cdots,\varepsilon_{[p/2]-1}$ in the above expression are multiplicatively independent in $\boldsymbol{C}^\times=\{\alpha\in\boldsymbol{C}:\alpha\neq 0\}$.*

The theorem is proven in [1] §3 (3.8) 2) for the case $k=3$. Then Prof. G. Fujisaki asked the author whether it is true for $k\geq 3$. In fact it is true as we see in this note. The author would like to express his gratitude to Prof. G. Fujisaki.

§2. **A sketch of the proof of the theorem.** For a fixed k, the sequence $P_p=P_{k,p}$, $p\geq 4$ of the polynomials satisfies the following recursion formula.

$$P_{p+2}(x)=(x^2+1)P_p(x)-x^2P_{p-2}(x) \qquad \text{for } p\geq 4. \tag{2.1}$$

Define new polynomials in $z=x+x^{-1}$ by,

$$Q_q(z):=x^{-q}P_{2q}(x) \qquad q=2,3,4,\cdots \tag{2.2}$$
$$R_q(z):=(x+1)^{-1}x^{-q}P_{2q+1}(x) \qquad q=2,3,4,\cdots.$$

Then the recursion formula (2.1) turns out to be,

$$Q_{q+1}(z)=zQ_q(z)-Q_{q-1}(z) \qquad q=2,\cdots \tag{2.3}$$
$$R_{q+1}(z)=zR_q(z)-R_{q-1}(z) \qquad q=2,\cdots.$$

Now let us show the following assertion.

Assertion. *The equation $Q_q(z)=0$ (resp. $R_q(z)=0$) has q real simple roots. $q-1$ of them lie in the interval $(-2,2)$ and the remaining one lies in the interval $(-\infty,-2)$.*