39. A Cyclic Vector in the Tensor Product of Irreducible Representations of Compact Groups

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1. Let G be a non-abelian connected compact Lie group and T a maximal torus in G with Lie algebras $\mathfrak{g}$, $\mathfrak{t}$ respectively. With respect to $\mathfrak{t}$, we introduce a lexicographic order on the set of roots of $\mathfrak{g}_c$ (the complexification of $\mathfrak{g}$). And we denote by X_k^+ ($k=1, 2, \cdots, n$) (resp. X_k^-) root vectors for all positive roots (resp. negative roots) in this order.

Any unitary representation U is canonically able to be extended to a representation $U(X)$ of $\mathfrak{g}_c$. When U is irreducible, we can define uniquely its highest weight μ as a linear form on $\sqrt{-1}\,\mathfrak{t}$. The highest (resp. the lowest) weight vector v in U is characterized up to constant as a vector satisfying $U(X_k^+)v=0$ (resp. $U(X_k^-)v=0$) for all k.

In [1] Theorem 3′, C. Fronsdal and T. Hirai proved the following

Theorem. *Let $v_1 \in E_1$* (resp. $v_2 \in E_2$) *be the non-zero highest* (resp. lowest) *weight vector for irreducible representation U_1* (resp. U_2) *of G. Then the vector $v_1 \otimes v_2$ in $E_1 \otimes E_2$ is a cyclic vector for the tensor product $U_1 \otimes U_2$.*

The purpose of this paper is to give another proof of this theorem.

2. Proof of Theorem. Since G is compact, we can assume that U_1, U_2 are unitary. And it is enough to show that for any irreducible component U in $U_1 \otimes U_2$ with representation space E in $E_1 \otimes E_2$,

(1) the vector $v_1 \otimes v_2$ is not orthogonal to E.

By weight vectors $v_j^\alpha \in E_j$ ($\alpha=1, 2, \cdots, m_j$) ($v_1=v_1^1, v_2=v_2^1$), any v in E is expanded in a unique way as

$$v=\sum_{\alpha,\beta} a(v, v_1^\alpha, v_2^\beta) v_1^\alpha \otimes v_2^\beta. \tag{2}$$

Especially the highest weight vector w in U is written as

$$w=\sum_{\alpha} v_1^\alpha \otimes u^\alpha \qquad (u^\alpha \in E_2), \tag{3}$$

here

$$u^\alpha=\sum_{\beta} a(w, v_1^\alpha, v_2^\beta) v_2^\beta.$$

The vector w satisfies for any k,

$$U(X_k^+)w=\sum_{\alpha} U_1(X_k^+)v_1^\alpha \otimes u^\alpha+\sum_{\alpha} v_1 \otimes U_2(X_k^+)u^\alpha=0. \tag{4}$$

Let the weight μ_1^γ be the highest among the set $\{\mu_1^\alpha\,;\, u^\alpha \neq 0 \text{ in } (4)\}$. Since the vector $U_1(X_k^+)v_1^\gamma$ has the weight higher than μ_1^γ, it must vanish for