19. *The Stokes Operator in L_r Spaces*

By Yoshikazu GIGA
Department of Mathematics, University of Tokyo

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Introduction. In this paper we shall report that the Stokes operator generates a bounded analytic semigroup of class C_0 in L_r spaces. Moreover, we shall decide domains of fractional powers of the Stokes operator. To show these we shall construct the resolvent of the Stokes operator, using pseudodifferential operators.

Let D be a bounded domain in $\boldsymbol{R}^n$ with the smooth boundary S. Let $1<r<\infty$ and let X_r be the closure in $(L_r(D))^n$ of all smooth solenoidal vector fields with compact supports in D. Then there exists the continuous projection P_r from $\boldsymbol{L}_r(D)=(L_r(D))^n$ onto X_r; see Fujiwara-Morimoto [5]. We denote by $W_r^m(D)$ the Sobolev space of order m. Set $\boldsymbol{W}_r^m(D)=(W_r^m(D))^n$. Then we define the Stokes operator by $A_r=-P_r\Delta$ ($\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$) whose domain is

$$D(A_r)=\{w \in \boldsymbol{W}_r^2(D)\cap X_r : w|_S=0\}.$$

Let $\varepsilon>0$, $\omega\geqq 0$ and let $\Sigma_{\varepsilon,\omega}$ denote the set of $\lambda \in \boldsymbol{C}$ such that $|\arg \lambda|\leqq \pi-\varepsilon$, $|\lambda|>\omega$. Then we have

Theorem 1. *For any $\varepsilon>0$ there exists a constant $C_{\varepsilon,r}$ independent of $f \in X_r$ and of $\lambda \in \Sigma_{\varepsilon,0}$ such that*

$$\|(\lambda+A_r)^{-1}f\|\leqq C_{\varepsilon,r}|\lambda|^{-1}\|f\|, \tag{1}$$

where $\|\ \|$ denotes the norm of $\boldsymbol{L}_r(D)$. Consequently, $-A_r$ generates a bounded analytic semigroup of class C_0 in X_r.

Remark. This result is partially known by Solonnikov [14]; he proved (1) for $|\arg \lambda|\leqq \delta+\pi/2$, where $\delta\geqq 0$ is small. Our result is new in the following two points:

i) We prove that the estimate (1) holds for larger domain of λ, that is, $\lambda \in \Sigma_{\varepsilon,0}$ for *any* positive ε.

ii) We construct the resolvent $(\lambda+A_r)^{-1}$ explicitly. This enables us to describe the domain of fractional power A_r^α of A_r. For the case of the Laplace operator the corresponding result is well known; see Fujiwara [4] and Seeley [12].

By Theorem 1 we can define A_r^α. Concerning A_r^α we have

Theorem 2. *For any $\varepsilon>0$ there exists a constant $M_{\varepsilon,r}$ independent of $f \in X_r$, $-1\leqq a<0$, $b \in \boldsymbol{R}$ such that*

$$\|A_r^{a+ib}f\|\leqq M_{\varepsilon,r}e^{\varepsilon|b|}\|f\|, \quad (i=\sqrt{-1}).$$

This implies, like Kato [6],

$$D(A_r^\alpha)=[X_r, D(A_r)]_\alpha, \qquad 0<\alpha<1, \tag{2}$$