12. *Class Number Calculation and Elliptic Unit. I*

Cubic Case

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Let K be a real cubic number field with the discriminant $D<0$. In the following, an effective algorithm will be given, to calculate the class number h and the fundamental unit ε_1 (>1) of K at a time.

Angell [1] has given a table of h and ε_1 of K for $D>-20000$. In the special case when $K=\boldsymbol{Q}(\sqrt[3]{m})$, a pure cubic number field, Dedekind [5] has given an analytic method to calculate h. In such a pure cubic case, Dedekind's method has been improved by several authors, see [3] and [13]. In all these algorithms, however, it is necessary to compute ε_1 by Voronoi's algorithm, see [6, pp. 232–290], before the calculation of h.

Our method does not need Voronoi's algorithm, and h and ε_1 are calculated at a time. The starting point of the method is the index formula on elliptic units given by Schertz, see [11] and [12], and the idea of the algorithm is learned from G. Gras and M.-N. Gras [8]. There is a similar algorithm to compute the class number and fundamental units of a real quartic number field which is not totally real and contains a quadratic subfield, see the author's [10]. The author expects that such an algorithm will be generalized to calculate the class number of a non-galois number field whose galois closure is an abelian extension over an imaginary quadratic number field.

§1. **Illustration of algorithm.** The class number h of K is given by the index of the subgroup generated by the so called "elliptic unit" η_e (>1) of K, of which the definition will be given in §4, in the group of positive units of K, see [11]:

(1) $$\eta_e=\varepsilon_1^h, \qquad \text{i.e. } h=(\langle\varepsilon_1\rangle:\langle\eta_e\rangle).$$

Our method consists of the following steps:

(i) to compute an approximate value of η_e (§4),

(ii) to compute the minimal polynomial of η_e over $\boldsymbol{Q}$ (Lemma 2),

(iii) for any unit $\xi(>1)$ of K, to give an explicit upper bound $B(\xi)$ of $(\langle\varepsilon_1\rangle:\langle\xi\rangle)$ (Proposition 1),

(iv) for any unit $\xi(>1)$ of K and for a natural number μ, to judge whether a real number $\sqrt[\mu]{\xi}\,(>1)$ is an element to K or not, and to compute the minimal polynomial of $\sqrt[\mu]{\xi}$ over $\boldsymbol{Q}$ if it is an element of K