11. *On Eisenstein Series for Siegel Modular Groups*

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Introduction. We present some results on Eisenstein series for Siegel modular groups. These results concern the action of Hecke operators and the Fourier coefficients. We refer to [3] for the motivation of these results. We use the notations of [4].

§ 1. Eisenstein series. For integers $n \geqq 0$ and $k \geqq 0$, we denote by $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$) the $\boldsymbol{C}$-vector space of all Siegel modular (resp. cusp) forms of degree n and weight k. (See [4, § 3] for Siegel modular forms of degree zero.) The space of Eisenstein series is $E_k(\Gamma_n) = S_k(\Gamma_n)^{\perp}$, which is the orthogonal complement of $S_k(\Gamma_n)$ in $M_k(\Gamma_n)$ with respect to the Petersson inner product $\langle\ ,\ \rangle$. For each even integer $k > 2n$, the space $E_k(\Gamma_n)$ is constructed from $M_k(\Gamma_{n-1})$ by using the Eisenstein series of Langlands [5] and Klingen [1]. To be precise we define a $\boldsymbol{C}$-linear map $[\ \ ]^{(n-r)} : M_k(\Gamma_r) \to M_k(\Gamma_n)$ for $0 \leqq r \leqq n$ and even $k > n+r+1$ as follows. Each modular form f in $M_k(\Gamma_r)$ is written uniquely as $f = \sum_{j=0}^{r} E_{r,j}^{k}(*, f_j)$ with cusp forms $f_j \in S_k(\Gamma_j)$ $(0 \leqq j \leqq r)$, where $E_{r,j}^{k}(*, f_j)$ is the Eisenstein series defined in Klingen [1]. We define $[f]^{(n-r)} = \sum_{j=0}^{r} E_{n,j}^{k}(*, f_j)$. Then $[f]^{(n-r)}$ is a modular form in $M_k(\Gamma_n)$ satisfying $\Phi^{n-r}([f]^{(n-r)}) = f$, where Φ is the Siegel operator. In particular, $[\ \ ]^{(0)}$ is the identity map, and we write $[\ \ ] = [\ \ ]^{(1)}$ for simplicity. Then it holds that

$$E_k(\Gamma_n) = [M_k(\Gamma_{n-1})] = \mathrm{Image}\,([\ \ ] : M_k(\Gamma_{n-1}) \to M_k(\Gamma_n))$$

for $n \geqq 1$ and even $k > 2n$. More precisely we have $E_k^r(\Gamma_n) = [S_k(\Gamma_r)]^{(n-r)}$ and $\oplus_{j=0}^{r} E_k^j(\Gamma_n) = [M_k(\Gamma_r)]^{(n-r)}$ for $0 \leqq r \leqq n$ and even $k > n+r+1$, where $E_k^j(\Gamma_n) = \mathfrak{S}_{kj}^{(n)}$ in the notation of Maass [6]. For $0 \leqq r \leqq n$ and even $k > 2n$, $[\ \ ]^{(n-r)}$ is the following $(n-r)$-times composition of $[\ \ ]$ ("$(n-r)$-th power"):

$$M_k(\Gamma_r) \xrightarrow{[\ ]} M_k(\Gamma_{r+1}) \xrightarrow{[\ ]} \cdots \xrightarrow{[\ ]} M_k(\Gamma_n).$$

We use also the following extended definition: if $f \in M_k(\Gamma_r)$, $r \leqq j \leqq n$, $k > n+r+1$ even, and $F = [f]^{(j-r)}$, then we define that $[F]^{(n-j)} = [f]^{(n-r)}$.

Theorem 1. *Let f be an eigen modular form in $M_k(\Gamma_r)$ for $r \geqq 0$ and even $k > n+r+1$ with $n \geqq r$. Then $[f]^{(n-r)}$ is an eigen modular form in $M_k(\Gamma_n)$.*

Proof. In this proof j runs over $j = 0, \cdots, r$. Write $f = \sum_j [f_j]^{(r-j)}$ with $f_j \in S_k(\Gamma_j)$, then $[f]^{(n-r)} = \sum_j [f_j]^{(n-j)} \in \oplus_{j=0}^{r} E_k^j(\Gamma_n)$. Take a Hecke