117. *An Estimate of the Roots of b-Functions by Newton Polyhedra*

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Introduction. In this note we give an estimate of the roots of b-functions of certain isolated singularities (Theorem 4.4).

The theory of b-functions and the proof given here are based on Yano [5]. In the real analytic case, the same estimate is given in Varchenko [4].

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§1. Let $\mathcal{O}$ be the set of germs of holomorphic functions at the origin O of C^{n+1}, $\mathcal{D}=\mathcal{O}[\partial/\partial x_0, \cdots, \partial/\partial x_n]$, $B_{pt}=D\delta$ where δ is the δ-function.

For any $f \in \mathcal{O}$, there exist $P(s) \in \mathcal{D}[s]$, $b(s) \in C[s]$ such that $P(s)f^{s+1} =b(s)f^s$ (Bernstein [1], Björk [2]). These $b(s)$ form an ideal and the generator of the ideal is called the b-*function* of f and denoted by $b_f(s)$. If $f(0)=0$, $b_f(s)$ is divided by $s+1$ and we put $\tilde{b}_f(s)=b_f(s)/(s+1)$. $\mathcal{J}_f(s) =\{P(s) \in \mathcal{D}[s] : P(s)f^s=0\}$.

Let $\Gamma_+(f)$ be the Newton polyhedron of f and $\{\gamma_1, \cdots, \gamma_m\}$ the set of all the n-dimensional faces of $\Gamma_+(f)$ not contained in $\{x : \prod_{i=0}^n x_i=0\}$, $\gamma_k=\{(x_0, \cdots, x_n) : \sum d_{k,i}x_i=1\}$. Then $d_k(x_i)=d_{k,i}$ defines a degree on $\mathcal{O}$, and we put $X_k=\sum d_{k,i}x_i\partial/\partial x_i$.

§2. From now on we assume that $f \in \mathcal{O}\,(f(0)=0)$ has an isolated singularity and is nondegenerate with respect to $\Gamma_+(f)$.

2.1. Theorem (Kashiwara-Yano). *α is a root of $\tilde{b}_f(s)$ if and only if there exists a nonzero element Δ of B_{pt} satisfying the following two conditions*:

(2.1.1) $\quad f(x)\Delta=0$ *and* $\partial f/\partial x_i\Delta=0, \quad i=0, \cdots, n,$

(2.1.2) $\quad$ *for any* $P(s) \in \mathcal{J}_f(s), \quad P(\alpha)\Delta=0.$

2.2. Theorem (Teissier [3]). *For any ideal I of $\mathcal{O}$, there exists $\nu_0 \in N$ such that, for any $\nu \in N$, $\overline{I^{\nu+\nu_0}}=I^\nu\cdot\overline{I^{\nu_0}}$, where $\bar{I}$ denotes the integral closure of I.*

2.3. Proposition. *Let $I=(x_0\partial f/\partial x_0, \cdots, x_n\partial f/\partial x_n)\mathcal{O}$. For any $\nu \in N$ and $g \in \mathcal{O}$, $g \in \overline{I^\nu}$ if and only if $d_k(g)\geqq\nu$, $k=1, \cdots, m$.*

§3. Construction of an operator $P(s) \in \mathcal{J}_f(s)$. An element of $\mathcal{D}[s]f^s$ is uniquely expressed as a finite sum $\sum_i a_i(x)f[i]$, $a_i \in \mathcal{O}$, $f[i]$