116. *Differentiability of Riemann's Function*

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1. Introduction. In this paper we discuss on the differentiability of the function

$$f(x)=\sum_{n=1}^{\infty} \sin n^2x/n^2.$$

Riemann proposed the problem that the function is nowhere differentiable, [2] and [8]. About the problem, J. P. Kahane [3] has investigated lacunary series. It was solved by J. Gerver [4] [5]. First G. H. Hardy [6] proved that the function is not differentiable at the point $\xi\pi$ where ξ is irrational or is a rational of the form $(2A+1)/2B$ or $2A/(4B+1)$. Later Gerver proved that $f(x)$ is differentiable at all points $(2A+1)\pi/(2B+1)$ with derivative $-1/2$, and not differentiable at the points $2A\pi/(2B+1)$.

The purpose of this paper is to give a shorter proof of the differentiability as well as a finer estimate of the function at points of rational multiple of π.

We states the following

Theorem 1. *The function*

$$F(x)=\sum_{n=1}^{\infty} \exp(in^2\pi x)/n^2\pi i$$

have the following behavior near $x=q/p$, *where* p *is a positive integer and* q *is an integer such that* q/p *is an irreducible fraction,*

$$\begin{aligned} &F(x+h)-F(x) \\ (1)\qquad &=R(p,q)p^{-1/2}\exp\left(\frac{\pi i}{4}\operatorname{sgn} h\right)|h|^{1/2}\operatorname{sgn} h-\frac{h}{2}+O(|h|^{3/2})\end{aligned}$$

as $h\to 0$ *where* $\operatorname{sgn} h=h/|h|$ *if* $h\neq 0$, $\operatorname{sgn} h=0$ *if* $h=0$, *and* $R(p,q)$ *is a constant defined by*

$$(2)\qquad R(p,q)=\begin{cases}\left(\dfrac{q}{p}\right)\exp\left(\dfrac{-\pi i}{4}(p-1)\right) & \text{if } p \text{ is odd and } q \text{ even,}\\ \left(\dfrac{p}{|q|}\right)\exp\left(\dfrac{\pi i}{4}q\right) & \text{if } p \text{ is even and } q \text{ odd,}\\ 0 & \text{if } p \text{ and } q \text{ are odd,}\end{cases}$$

with the Jacobi's symbol (p/q) (*see* [7]).

This theorem easily shows Gerver's results and gives a finer estimate of $f(x)$ at the points of rational multiple of π. The author wishes to thank Prof. Jean Pierre Kahane for helpful suggestions.