10. *On Siegel Eigenforms*

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Introduction. We note some properties of Siegel eigenforms of general degree relating to the algebra of Hecke operators. We refer to [5]–[7] for motivations and examples. (These examples satisfy the "multiplicity one conjecture".)

§ 1. Eigencharacters. For integers $n \geqq 1$ and $k \geqq 0$, we denote by $M_k(\Gamma_n)$ the vector space over the complex number field $\boldsymbol{C}$ consisting of all Siegel modular forms of degree n and weight k. The space of cusp forms is $S_k(\Gamma_n) = \mathrm{Ker}\,(\Phi: M_k(\Gamma_n) \to M_k(\Gamma_{n-1}))$, where Φ is the Siegel operator. As usual we understand that $M_k(\Gamma_0) = S_k(\Gamma_0) = \boldsymbol{C}$ for $k \geqq 0$; see § 3 for Siegel modular forms of degree zero. For each integer $n \geqq 1$, we denote by $L = L^{(n)}$ the abstract Hecke algebra of degree n over $\boldsymbol{C}$ as in Andrianov [1, § 1.3]. For each integer $k \geqq 0$, we denote by $\tau = \tau_k^{(n)}: L \to \mathrm{End}_{\boldsymbol{C}}(M_k(\Gamma_n))$ the representation of L on $M_k(\Gamma_n)$ defined in Andrianov [1, (1.3.3)]. We denote by $\boldsymbol{T} = \boldsymbol{T}(M_k(\Gamma_n)) = \tau(L)$ the $\boldsymbol{C}$-algebra of all Hecke operators on $M_k(\Gamma_n)$. We put $\hat{\boldsymbol{T}} = \mathrm{Hom}_{\boldsymbol{C}}(\boldsymbol{T}, \boldsymbol{C})$ ($\boldsymbol{C}$-algebra homomorphisms), and for each $\lambda \in \hat{\boldsymbol{T}}$ we put $M_k(\Gamma_n; \lambda) = \{f \in M_k(\Gamma_n) \mid Tf = \lambda(T)f \text{ for all } T \in \boldsymbol{T}\}$ and $m(\lambda) = \dim_{\boldsymbol{C}} M_k(\Gamma_n; \lambda)$ the "multiplicity" of λ. We denote by $\Lambda(\boldsymbol{T}) = \{\lambda \in \hat{\boldsymbol{T}} \mid m(\lambda) \geqq 1\}$ the set of all "eigencharacters" of $\boldsymbol{T}$. Then ${}^{\#}\Lambda(\boldsymbol{T}) \leqq \dim_{\boldsymbol{C}} M_k(\Gamma_n)$, where ${}^{\#}\Lambda(\boldsymbol{T})$ denotes the cardinality of $\Lambda(\boldsymbol{T})$. A formulation of the "multiplicity one conjecture" is that $m(\lambda) = 1$ for all $\lambda \in \Lambda(\boldsymbol{T})$. This is equivalent to the following equality ${}^{\#}\Lambda(\boldsymbol{T}) = \dim_{\boldsymbol{C}} M_k(\Gamma_n)$, since we have $M_k(\Gamma_n) = \oplus_\lambda M_k(\Gamma_n; \lambda)$ and $\dim_{\boldsymbol{C}} M_k(\Gamma_n) = \sum_\lambda m(\lambda)$ where λ runs over $\Lambda(\boldsymbol{T})$. We say that a modular form f in $M_k(\Gamma_n)$ is an eigen modular form (or "eigenform") if f is a non-zero modular form belonging to $M_k(\Gamma_n; \lambda)$ for a $\lambda \in \Lambda(\boldsymbol{T})$. Such a λ is uniquely determined by f, and we denote it by $\lambda(f)$. In this case we denote by $\lambda(T, f)$ the value of $\lambda(f)$ at $T \in \boldsymbol{T}$: $Tf = \lambda(T, f)f$. We write $\lambda(m, f) = \lambda(T(m), f)$ for each integer $m \geqq 1$, where $T(m) \in \boldsymbol{T}$ is the Hecke operator studied by Maass [8] normalized as in Andrianov [1, (1.3.15)].

The Fourier expansion of a modular form f in $M_k(\Gamma_n)$ is denoted by $f = \sum_{T \geqq 0} a(T, f) q^T$ with $q^T = \exp(2\pi\sqrt{-1} \cdot \mathrm{trace}\,(TZ))$ where Z is a variable on the Siegel upper half space of degree n and T runs over all $n \times n$ symmetric semi-integral positive semi-definite matrices. Let R be a subring of $\boldsymbol{C}$. We put $M_k(\Gamma_n)_R = \{f \in M_k(\Gamma_n) \mid a(T, f) \in R \text{ for all } T \geqq 0\}$ (an R-module) and $M_k(\Gamma_n; \lambda)_R = M_k(\Gamma_n; \lambda) \cap M_k(\Gamma_n)_R$ (an R-