105. A Characterization of Smooth Banach Spaces

By Bruce CALVERT*) and Simeon REICH**)

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1981)

Our main purpose in this note is to show how (nonlinear) accretive operators can be used to characterize smooth (reflexive) Banach spaces. More precisely, we show that if a Banach space E is not smooth, then there is accretive $A\subset E\times E$ that satisfies the range condition such that $\mathrm{cl}\,(R(A))$ does not have the minimum property (see the definitions below). On the other hand, it is also true that if a reflexive space E is smooth and $A\subset E\times E$ is an accretive operator that satisfies the range condition, then $\mathrm{cl}\,(R(A))$ has the minimum property. Consequently, a reflexive Banach space E is smooth if and only if $\mathrm{cl}\,(R(A))$ has the minimum property for all accretive $A\subset E\times E$ that satisfy the range condition (Theorem 1). In fact, the same result is true if A is restricted to be of the form $I-T$, where T is nonexpansive. This provides an answer to a question of Pazy [3]. In addition, we characterize (finite-dimensional) smooth Banach spaces by using an invariance criterion for nonlinear semigroups (Theorem 2).

Let E be a real Banach space, and let I denote the identity operator. Recall that a subset A of $E\times E$ with domain $D(A)$ and range $R(A)$ is said to be *accretive* if $|x_1-x_2|\le|x_1-x_2+r(y_1-y_2)|$ for all $[x_i, y_i]\in A$, $i=1, 2$, and $r>0$. The resolvent $J_r: R(I+rA)\to D(A)$ of A is defined by $J_r=(I+rA)^{-1}$. We denote the closure of a subset D of E by $\mathrm{cl}\,(D)$, its closed convex hull by $\mathrm{clco}\,(D)$ and its distance from a point x in E by $d(x, D)$. We shall say that A satisfies the *range condition* if $R(I+rA)\supset\mathrm{cl}\,(D(A))$ for all $r>0$. In this case, $-A$ generates a nonexpansive nonlinear semigroup $S:[0,\infty)\times\mathrm{cl}\,(D(A))\to\mathrm{cl}\,(D(A))$ by the exponential formula: $S(t)x=\lim_{n\to\infty}(I+(t/n)A)^{-n}x$. A closed subset D of E is said to have the *minimum property* [3] if $d(O, \mathrm{clco}\,(D))=d(O, D)$.

Recall that the norm of E is said to be Gâteaux differentiable (and E is said to be *smooth*) if $\lim_{t\to 0}(|x+ty|-|x|)/t$ exists for each x and y in $U=\{x\in E:|x|=1\}$. The duality map from E into the family of nonempty subsets of its dual E^* is defined by $J(x)=\{x^*\in E^*:(x, x^*)=|x|^2=|x^*|^2\}$. It is single-valued if and only if E is smooth. An

*) Department of Mathematics, The University of Auckland, Auckland, New Zealand.

**) Department of Mathematics, The University of Southern California, Los Angeles, California 90007.