99. *On Hilbert Modular Forms*

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Introduction. In the theory of elliptic modular forms, it is known that every modular form whose Fourier coefficients lie in $\boldsymbol{Z}[1/6]$ is an isobaric polynomial in E_4 and E_6 with coefficients in $\boldsymbol{Z}[1/6]$, where E_4 and E_6 are the normalized Eisenstein series of respective weights four and six.

In this paper, we give an analogous result for Hilbert modular forms for the real quadratic field $K=\boldsymbol{Q}(\sqrt{5})$. Namely, we show that every symmetric Hilbert modular form for K whose Fourier coefficients lie in $\boldsymbol{Z}[1/2]$ can be represented as an isobaric polynomial in certain forms X_2, X_6 and X_{10} with coefficients in $\boldsymbol{Z}[1/2]$.

§ 1. Hilbert modular forms for $\boldsymbol{Q}(\sqrt{5})$. Let $\mathfrak{o}_K$ be the ring of integers in $K=\boldsymbol{Q}(\sqrt{5})$. Let $\boldsymbol{H}$ denote the upper half-plane. Put $\Gamma_K=SL(2,\mathfrak{o}_K)$ and for an element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ_K, we put $\gamma^*=\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$ where the star denotes the conjugation in K.

We let Γ_K operate on $\boldsymbol{H}^2=\boldsymbol{H}\times\boldsymbol{H}$ by:

$$\gamma\cdot(z_1,z_2)=(\gamma z_1,\gamma^* z_2)=\left(\frac{az_1+b}{cz_1+d},\frac{a^*z_2+b^*}{c^*z_2+d^*}\right),\qquad (z_1,z_2)\in\boldsymbol{H}^2.$$

Further, for any $\tau=(z_1,z_2)\in\boldsymbol{H}^2$ and $\nu\in K$, we put

$$N(\nu\tau)=\nu z_1\cdot\nu^* z_2,\qquad tr(\nu\tau)=\nu z_1+\nu^* z_2.$$

A holomorphic function $f(\tau)$ on $\boldsymbol{H}^2$ is called a *symmetric Hilbert modular form of weight* k if it satisfies the following conditions:

(1) For every element $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Γ_K, $f(\tau)$ satisfies a functional equation of the form

$$f(\gamma\cdot\tau)=N(c\tau+d)^k f(\tau);$$

(2) $f((z_1,z_2))=f((z_2,z_1))$.

The set of such functions forms a complex vector space $A_{\boldsymbol{C}}(\Gamma_K)_k$. Any element $f(\tau)$ in $A_{\boldsymbol{C}}(\Gamma_K)_k$ admits a Fourier expansion of the form

$$f(\tau)=\sum_{\substack{\nu\equiv 0 \bmod (1/\sqrt{5})\\ \nu\gg 0 \text{ or } 0}} a_f(\nu)\exp\,[2\pi i tr(\nu\tau)],$$

where the sum extends over all totally positive numbers ν in K satisfying $\nu\equiv 0 \bmod (1/\sqrt{5})$.

For a subring R of $\boldsymbol{C}$, we put

$$A_R(\Gamma_K)_k=\{f\in A_{\boldsymbol{C}}(\Gamma_K)_k \mid a_f(\nu)\in R \text{ for all } \nu\equiv 0\ (1/\sqrt{5}),\ \nu\gg 0 \text{ or } 0\}.$$