97. On the Wiener-Schoenberg Theorem for Asymptotic Distribution Functions

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The fundamental theorem of H. Weyl [6] concerning the theory of uniform distribution mod 1 was generalized by E. Hlawka [1] and G. M. Petersen [3] to the case of almost convergence and by M. Tsuji [5] to that of weighted means. Also the study of asymptotic distribution functions mod 1 was initiated by Schoenberg [4]. He obtained the condition under which a sequence should have the asymptotic distribution mod 1.

In this note we shall unify the concepts to show theorems related to the theorem of Schoenberg:

A sequence (x_n) of real numbers has the a.d.f. (mod 1) *if and only if for every positive integer h the limit*

$$\alpha_h = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n}$$

exists and, in addition

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} |\alpha_h|^2 = 0.$$

This is also obtained by Wiener [7] in a slightly different form. Now we shall begin with two key definitions:

Let f be a complex-valued continuous function on $(-\infty, +\infty)$ with period 1.

Definition 1. The sequence (x_n) is said to have the (M, λ_n)-asymptotic distribution function mod 1 (abbreviated (M, λ_n)-*a.d.f.* (mod 1)) $g(x)$ if

$$(1) \qquad \lim_{n\to\infty} \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k f(x_k) = \int_0^1 f(x)dg(x),$$

where

$$\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \quad \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq \cdots > 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty.$$

Definition 2. The sequence (x_n) is said to have the (M, λ_n)-asymptotic well-distribution function mod 1 (abbreviated (M, λ_n)-*a.w.d.f.* (mod 1)) $g(x)$ if

$$(2) \qquad \lim_{n\to\infty} \frac{1}{\Lambda_n^k} \sum_{\nu=k+1}^{k+n} \lambda_\nu f(x_\nu) = \int_0^1 f(x)dg(x) \quad \text{uniformly in } k=0,1,2,\cdots,$$

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