90. *Vertex Operators and τ Functions*

Transformation Groups for Soliton Equations. II

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It was E. Galois who noticed the importance of the transformation groups of algebraic equations. This idea has been powerful in several fields. The theory of Picard-Vessiot, for example, is to study linear ordinary differential equations by the aid of Lie groups acting on them.

The purpose of this article is to add another example to the philosophy of E. Galois by studying soliton-type equations such as the Korteveg de Vries equation (KdV equation), the Boussinesq equation, the Kadomtsev-Petviashvili equation (KP equation), etc. through their transformation groups. In these cases, the groups are not finite-dimensional any more, but infinite-dimensional Lie groups or the groups associated with so called Kac-Moody Lie algebras.

Lepowsky-Wilson [1] constructed an irreducible representation of the affine Lie algebra $A_1^{(1)}$, the central extension of $sl(2; C[t, t^{-1}])$, on the infinite-dimensional vector space $C[x_1, x_3, x_5, \cdots]$, which has the constant function as a highest weight vector. In their construction, the "vertex operator"

$$X(p)=\exp\left(2\sum x_j p^j\right)\exp\left(-2\sum \frac{1}{j}\frac{\partial}{\partial x_j}p^{-j}\right) \qquad (j \text{ odd}>0)$$

plays a crucial role. We noticed that this vertex operator is nothing but the infinitesimal Bäcklund transformation for the KdV equation in soliton theory. Thus, the group $A_1^{(1)}$ is the transformation group of the hierarchy of the KdV equations and the space of associated τ functions coincides with the orbit of the highest weight vector. This fact is also true for other soliton-type equations. In the case of the KP equation, the Lie algebra $\mathfrak{gl}(\infty)$ operates on $C[x_1, x_2, \cdots]$ through the vertex operator and the space of τ functions coincides with the orbit of the constant function. This corresponds to the remarkable discovery of M. Sato and Y. Sato [2] that the space of τ functions of the KP equation is the infinite-dimensional Grassmann manifold.

The hierarchy of the KdV equation is subholonomic in the sense that the general solution depends on finitely many arbitrary functions

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