86. On Richardson Classes of Unipotent Elements in Semisimple Algebraic Groups

By Takeshi HIRAI
Department of Mathematics, Kyoto University

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Let G be a connected semisimple algebraic group over an algebraically closed field K of characteristic zero. Let P be a parabolic subgroup of G and U_P its unipotent radical. A unipotent class in G is called a Richardson class corresponding to P if it intersects U_P densely [6]. We study here the correspondence between Richardson classes and parabolic subgroups in detail. Note here that, as is shown in [3], we naturally encounter the notion of Richardson classes or more generally that of induced classes, in the study of Fourier transform of unipotent orbital integrals on a connected semisimple Lie group.

§1. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Σ the root system of $(\mathfrak{g}, \mathfrak{h})$, Σ^+ the set of all positive roots in Σ, and Π the set of all simple roots. For a subset Γ of Π, we define a standard parabolic subgroup $P(\Gamma)$ of G as follows. Put $\Gamma^\perp=\{X \in h ; \gamma(X)=0 \ (\gamma \in \Gamma)\}$, and $L(\Gamma)=\{g \in G ; \mathrm{Ad}(g)X=X \ (X \in \Gamma^\perp)\}$. Further let $\langle \Gamma \rangle$ be the set of all roots in Σ expressed as integral linear combinations of $\gamma \in \Gamma$, U_α the one-dimensional unipotent subgroup corresponding to $\alpha \in \Sigma$, and $U(\Gamma)$ the subgroup generated by U_α's for $\alpha \in \Sigma^+ - \langle \Gamma \rangle$. Then $P(\Gamma)=L(\Gamma)U(\Gamma)$ is a parabolic subgroup of G with a Levi subgroup $L(\Gamma)$ and the unipotent radical $U(\Gamma)$. Note that $\langle \Gamma \rangle$ is the root system of $L(\Gamma)$ and $\Sigma^+ - \langle \Gamma \rangle$ is the ideal $I(\Pi - \Gamma)$ of Σ^+ generated by $\Pi - \Gamma$ in the sense of [7, §2].

For $\Gamma, \Gamma' \subset \Pi$, we define "$\Gamma \sim \Gamma'$ *in* Σ" if the Richardson classes corresponding to $P(\Gamma)$ and $P(\Gamma')$ coincide with each other. Remark that $\Gamma \sim \Gamma'$ here is equivalent to $I(\Pi - \Gamma) \sim I(\Pi - \Gamma')$ in [7, §2].

Rewriting Theorem 1.7 in [5], we have the following.

Theorem 1 [5]. *Let W be the Weyl group of the root system Σ, and let $\Gamma, \Gamma' \subset \Pi$. If $w\Gamma = \Gamma'$ for some $w \in W$, then $\Gamma \sim \Gamma'$.*

We also have the following general theorem.

Theorem 2. *Let Π_1, Π_2 be two subsets of Π orthogonal to each other, and let Γ_i, Γ'_i be two subsets of Π_i for $i=1, 2$. Assume that $\Gamma_i \sim \Gamma'_i$ in the root system $\langle \Pi_i \rangle$ for $i=1, 2$. Then $\Gamma=\Gamma_1 \cup \Gamma_2 \sim \Gamma'=\Gamma'_1 \cup \Gamma'_2$ in $\Sigma=\langle \Pi \rangle$. Here $\Pi_2=\phi$ may be admitted.*

§2. We call a subsystem of the relations $\Gamma \sim \Gamma'$ in various (Π, Σ)