78. On the Regularity of Arithmetic Multiplicative Functions. III

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(Communicated by Shokichi IYANAGA, M. J. A., June 11, 1981)

We present some new results concerning multiplicative functions.

1. Statement of results. Theorem. *Let $f(n)$ be a multiplicative arithmetic function. Suppose that there exists a positive non-decreasing function $g(x)$ such that*

i) $\lim_{x\to\infty} g(dx)/g(x)=h(d)$ *exists for any* $d\in N$,

ii) $\limsup_{x\to\infty} \frac{1}{g(x)} \sum_{n\leqslant x} |f(n+1)-f(n)|=0.$

a) *If*

iii) $\limsup_{x\to\infty} \frac{1}{g(x)} \left|\sum_{n\leqslant x} f(n)\right|>0,$

then, $f(n)$ is completely multiplicative, and there exists $\lambda\geqslant -1$ such that $|f(n)|=n^{\lambda}$.

b) *If*

iii)′ $\lim_{x\to\infty} \frac{1}{g(x)} \sum_{n\leqslant x} f(n)=M$ *exists and* $M\neq 0$,

then there exists $\lambda\geqslant -1$ such that $f(n)=n^{\lambda}$.

2. Sketch of proof of the theorem. We deduce from assumptions i)–iii) by partial summation that, for any $d\in N$,

$$(*)\qquad \left|\sum_{\substack{n\leqslant x\\ d|n}} f(n)-\frac{1}{d}\sum_{n\leqslant x} f(n)\right|=o(g(x)).$$

We can prove easily from here that $f(n)\neq 0$ for any $n\in N$. In fact, for any prime p and any positive integer r, we have

$$\left|f(p^r)\sum_{\substack{n\leqslant xp^{-r}\\ (n,p)=1}} f(n)-\left(1-\frac{1}{p}\right)\frac{1}{p^r}\sum_{n\leqslant x} f(n)\right|$$
$$=\left|\sum_{\substack{n\leqslant x\\ p^r|n}} f(n)-\frac{1}{p^r}\sum_{n\leqslant x} f(n)-\sum_{\substack{n\leqslant x\\ p^{r+1}|n}} f(n)+\frac{1}{p^{r+1}}\sum_{n\leqslant x} f(n)\right|=o(g(x)).$$

On the other hand, condition iii) gives

$$\limsup_{x\to\infty} \frac{1}{g(x)}\left(1-\frac{1}{p}\right)\frac{1}{p^r}\left|\sum_{n\leqslant x} f(n)\right|>0,$$

and consequently $f(p^r)\neq 0$ for any p and any r. Then we can prove the *complete multiplicativity* of $f(n)$, by means of the same method as

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