6. *The Lax-Milgram Theorem for Banach Spaces. II*

By S. RAMASWAMY
School of Mathematics, Tata Institute of Fundamental Research

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1981)

§ 0. This paper is a sequel to [1] wherein we proved the Lax-Milgram theorem for a continuous and coercive bilinear form on a Banach space over $\boldsymbol{R}$. Here, we deal with the question of how far coercivity is necessary for the validity of the Lax-Milgram theorem. We construct a counter-example to show that coercivity is not necessary even on Hilbert spaces for the validity of the Lax-Milgram theorem. However, we prove that it is necessary in case the bilinear form 'a' is symmetric and positive-definite in the sense that $a(x, x)>0$ $\forall x \neq 0$.

The counter-example was constructed for me by Dr. V.S. Sunder. It is a pleasure for me to thank him for his help.

§ 1. Let V be a normal space over $\boldsymbol{R}$. Let $\|x\|$ denote the norm of the element $x \in V$. Let V' be the dual of V. Let 'a' be a continuous bilinear form on V. We do not necessarily assume that $\mathrm{a}(x, x)>0$ $\forall x \neq 0$.

We have the maps A and B from V to V' defined as $Ax(y)=a(y, x)$ and $Bx(y)=a(x, y)$. A and B are both continuous from V to V'. Let A^*(resp. B^*) be the adjoint of A(resp. B). A^* and B^* are maps from V'' (the double dual of V) to V'. It is easily seen that B (resp. A) is the restriction of A^*(resp. B^*) to V.

Motivated by the Lax-Milgram [1], we make the following definition.

Definition 1. V is said to have the *right* (resp. *left*) *Lax-Milgram property* with respect to 'a' if $\forall f \in V'$, $\exists$ a unique $u \in V$ such that $f(v)=a(v, u)$ (resp. $f(v)=a(u, v)$) $\forall v \in V$.

When 'a' is symmetric, it is clear that to say V has the right Lax-Milgram property is the same as to say that V has the left Lax-Milgram property. In this case, we simply speak of "*the Lax-Milgram property*".

The definition states that V has the right (resp. left) Lax-Milgram property with respect to 'a' iff A(resp. B) is one-one and onto, i.e. iff A(resp. B) is an isomorphism of V and V' in the algebraic sense.

Definition 2. A bilinear form 'a' on V is said to be *non-degenerate* if $\forall y \neq 0$, $\exists x, z \in V$ such that $a(x, y) \neq 0$ and $a(y, z) \neq 0$.

If 'a' has the property that $a(x, x)>0$ $\forall x \neq 0$, then it is clearly non-