55. A Remark on the Completeness of the Bergman Metric

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§1. **Introduction.** The purpose of this note is to prove the following theorem by modifying the argument in a recent work of P. Pflug (cf. [4]).

Theorem 1. *Let D be a bounded pseudoconvex domain in $\boldsymbol{C}^n$ with a C^1-smooth boundary. Then D is complete with respect to*

$$d_D := \sum_{\alpha,\beta} \frac{\partial^2 \log K(z\,;D)}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha d\bar{z}^\beta.$$

Here we put $z=(z^1,\cdots,z^n)$ and denote by $K(z\,;D)$ the Bergman kernel function of D.

The metric d_D was first introduced by S. Bergman [1], and S. Kobayashi [2] asked "Which bounded domain (of holomorphy) in $\boldsymbol{C}^n$ is complete with respect to d_D?"

The author is grateful to Prof. P. Pflug who informed him of his very interesting result.

§2. **Preliminaries.** We put

$$K(z,w\,;D) := \sum_{i=1}^{\infty} f_i(z)\overline{f_i(w)},$$

where $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis of $L_h^2(D) := \{f\,;$ holomorphic, square integrable on $D\}$.

Lemma 1 (cf. Lemma 3 in [4, IV]). *Assume the sequence $\{z_\nu\}_{\nu=1}^{\infty} \subset D$ to be a Cauchy-sequence with respect to d_D. Then there exist a subsequence $\{z_{\nu(u)}\}_{u=1}^{\infty}$ and real numbers θ_u such that the sequence*

$$\left\{\frac{K(\ ,z_{\nu(u)}\,;D)}{K(z_{\nu(u)},z_{\nu(u)}\,;D)^{1/2}}e^{i\theta_u}\right\}_{u=1}^{\infty}$$

is a Cauchy-sequence in $L_h^2(D)$ whose members are all of modulus one.

From Lemma 1 we can deduce the following

Lemma 2. *Assume that d_D is not complete. Then there is a sequence $\{z_\nu\}_{\nu=1}^{\infty} \subset D$ converging to a point $z^* \in \partial D$, such that*

$$\lim_{\min\{\nu,\mu\}\to\infty} \left(1-\frac{|K(z_\nu,z_\mu\,;D)|}{K(z_\nu,z_\nu\,;D)^{1/2}K(z_\mu,z_\mu\,;D)^{1/2}}\right)=0.$$

Proof. We only have to note that

$$\left(\frac{K(\ ,z_\nu\,;D)}{K(z_\nu,z_\nu\,;D)^{1/2}}, \frac{K(\ ,z_\mu\,;D)}{K(z_\mu,z_\mu\,;D)^{1/2}}\right)=\frac{K(z_\mu,z_\nu\,;D)}{K(z_\nu,z_\nu\,;D)^{1/2}K(z_\mu,z_\mu\,;D)^{1/2}}.$$

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