54. *Eta-Function on S^3*

By Kiyoshi KATASE
Department of Mathematics, Gakushuin University

(Communicated by Kunihiko KODAIRA, M. J. A., April 13, 1981)

Let M be a compact oriented riemannian manifold of dimension $2q-1$, $\bigwedge^p(M)$ be the vector space over $\boldsymbol{R}$ of all differential p-forms on M and put $\bigwedge^{ev}(M)=\sum_{i=0}^{q-1}\bigwedge^{2i}(M)$. Let $A:\bigwedge^{ev}(M)\to\bigwedge^{ev}(M)$ be a self-adjoint elliptic first order differential operator defined by

$$A\phi=(\sqrt{-1})^q(-1)^{p+1}(*d-d*)\phi$$

where the degree of ϕ is equal to $2p$, d is the exterior differential and $*$ is the Hodge duality operator. Note that the square $A^2=A\circ A$ is the Laplace operator $\Delta=d\delta+\delta d$, where δ is the codifferential of d.

When a compact group G acts on M, M. F. Atiyah, V. K. Patodi and I. M. Singer [1] defined a function

$$\eta_A(g,s)=\sum_{\lambda\neq 0}(\operatorname{sign}\lambda)Tr(g\,|\,E_\lambda)\cdot|\lambda|^{-s}$$

for g in G, where the summation is taken over all distinct eigenvalues of A and $g\,|\,E_\lambda$ is the transformation induced by g on the λ-eigenspace E_λ. They also showed as an example that if M is the circle S^1 and g is rotation through an angle α, then

$$\eta_A(g,s)=2\sqrt{-1}\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n^s}.$$

But it seems to be very hard to obtain directly this function for other manifolds. J. J. Millson [6] showed a method to calculate this eta-function on homogeneous spaces by using the Selberg zeta-function.

In this paper we show how to calculate more elementarily when M is the 3-sphere S^3 and

$$g=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos\varphi & -\sin\varphi\\ 0 & 0 & \sin\varphi & \cos\varphi\end{bmatrix}\in SO(4)$$

by determining the basis for the eigenspace of A. Our result is the following equation:

$$\eta_A(g,s)=-\frac{2\sin\varphi}{\cos\theta-\cos\varphi}\sum_{k=0}^{\infty}\frac{\sin(k+2)\theta}{(k+2)^s}-\frac{2\sin\theta}{\cos\varphi-\cos\theta}\sum_{k=0}^{\infty}\frac{\sin(k+2)\varphi}{(k+2)^s}.$$

Details and further arguments will be given elsewhere.

1. Let $H_k(\boldsymbol{R}^{n+1})$ be the vector space over $\boldsymbol{R}$ consisting of all harmonic homogeneous polynomials of degree k on $\boldsymbol{R}^{n+1}$ and let $H_k^p(\boldsymbol{R}^{n+1})$ be the vector space over $\boldsymbol{R}$ consisting of all p-forms on $\boldsymbol{R}^{n+1}$ of which